

Correction of systematic errors in a deductive method for determining the uncertainty of results obtained from processing measurement data series

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ABSTRACT

Systematic error correction remains a fundamental problem in measurement science, particularly when the final result is obtained by processing a series of raw measurement data. The main aim of this article is to demonstrate that the developed deductive method for determining measurement result uncertainty does not require correction of systematic errors in the raw results, because, under the assumptions of this method, expressing the final result as a measurand interval removes the need for separate systematic error correction. An additional aim of this work is to present the deductive method in the form of a coherent set of definitions and expressions that can be used as a basis for algorithmic implementation of the procedures for determining uncertainty and performing metrological analysis of measurements conducted in both industrial settings and research laboratories. Analytical considerations are illustrated with examples, including a description of a Monte Carlo experiment demonstrating the application of the proposed method to the correction of systematic error in a result obtained by processing a measurement data series.

Keywords: measurement error, systematic error correction, measurand, uncertainty, measurement data processing algorithm.

INTRODUCTION

The deductive method is a coherent set of definitions and mathematical expressions used for the metrological analysis of the measurement process and, ultimately, for determining an interval description of its result. The foundations of the mathematical formalism used in this work were developed for the purpose of reconstructing static and dynamic signals [1], but the formalism required supplementation before it could be universally applied in procedures for determining the uncertainty of measurement results. Thus, in [2] the concept of a measurand model was introduced, which allowed the developed formalism to be applied to determine the uncertainty of output results of linear and nonlinear measurement data processing algorithms. In turn, in [3]

the measurand was defined in a more general sense than in the dictionary [4] as a manifestation (sample) of the measured quantity. This made it possible to define the measurement window as a sequence of measurands of the same time-varying quantity and then to introduce the division of errors into types A and B, which is crucial for describing error propagation through the algorithms. This article introduces a definition of systematic error in accordance with the general definition of measurement error derived from an analysis of the measurement process using a standard with a quantum (granular) structure [1] and thus completes the process of building the formal apparatus referred to here as the deductive method for determining the uncertainty of a measurement result. The basic concept underlying the deductive method is to express the measurement result

in interval form. This concept is consistent with the assumptions of the international metrological organizations OIML and BIPM, as described in several documents entitled Guide to the Expression of Uncertainty in Measurement (GUM). The latest of these documents [5] is a compendium of recommendations for determining the uncertainty of measurement results. These recommendations are referred to here as the GUM method, and their comprehensive analysis can be found in the doctoral thesis [6]. However, the formal apparatus used in the deductive method differs significantly from that used in the GUM method because the two methods are based on different concepts. The basis of the deductive method is the definition of the measurement error, while the basic concept of the GUM method is uncertainty, defined as “a non-negative parameter characterizing the dispersion of the quantity values attributed to a measurand, based on the information used” [4]. As a consequence, the total uncertainty of the measurement result obtained according to the GUM method is calculated by combining partial uncertainties, while in the deductive method, the uncertainty is a parameter of the uncertainty interval determined on the basis of the distribution of the total measurement error obtained as the sum of random partial errors. The definition of systematic error used in the deductive method allows calculations to be performed on raw measurement results, as this error is eliminated from the final result by combining the uncertainty interval and the measurand estimate. Therefore, in this case, we can speak of “automatic” correction of the systematic error in the final result. This important property of the deductive method makes it possible to avoid correcting systematic errors in partial results, which is quite a complex task, especially when dealing with large volumes of measurement data affected by multiple partial errors. Correction of systematic errors, understood as their removal from the raw measurement results, is widely used in every field involving measurement. It is implemented in a manner specific to each research discipline and each industrial measurement application. In addition to the widely applied correction in accordance with the GUM method [5], Kalman filter-based approaches [7] and artificial neural network-based compensation methods [8] have also been proposed for this purpose. A description of the use of Bayesian methods to solve problems related to systematic errors can be found in [9], and the correction of

these errors in the frequency domain is analyzed in [10]. Reference [11] is noteworthy because it describes a number of universal procedures used for this purpose in scientific research. These procedures include the identification and modeling of systematic errors, as well as methods for correcting them. The deductive method provides the means to implement all three procedures. Systematic error correction is a necessary element of the process of determining the uncertainty and obtaining the estimate, i.e., the number that best approximates the true value of the measured quantity. Systematic error is defined in the vocabulary [4] as ‘component of measurement error that in replicate measurements remains constant or varies in a predictable manner’. This definition is useful when measurements are performed in laboratory settings that allow repeated measurements of the same quantity under the same measurement conditions [12]. However, it cannot be applied to industrial measurements in which quantities change over time and their instantaneous values can only be measured once. The deductive method avoids this problem through automatic correction of systematic error under all conditions that can be described by the measurement process model. This is due to a modification of the concept of the measurand as a manifestation (sample) of the measured quantity [3], which allows the basic component of the systematic error to be defined as the expected value of the probability distribution describing the measurement error of the measurand. This means that a random error without such a systematic component has an expected value of zero, and an error with a standard deviation of zero represents only a systematic error.

SYSTEMATIC ERROR OF A SINGLE MEASUREMENT RESULT

The mathematical apparatus used in the deductive method is based on the definition of the measurement error using a quantum (granular) standard. Such standards are used in A/D converters, which can both provide single results and serve as sources of data sequences for processing algorithms. The analysis of the quantization process [1] shows that, after measurement, the result is affected by the error

$$e = x - \tilde{x}, \quad (1)$$

defined as the difference between the unknown true value of the measurand x and the result $\tilde{x}$ of its measurement. If no operations have been performed to remove systematic errors from the measurement result, the result is called the raw or uncorrected result [4].

Equation 1 is consistent with the general definition of error given in the vocabulary [4], which defines error as the difference between two quantities, the second of which is the reference quantity. In the case of definition (1), the reference quantity is the measurement result, which is quite obvious, as it is the only number known after the measurement. However, it should be emphasized that this definition differs significantly from the definition of error given in [4]. Measurement error results from many factors that disturb the measurement process or otherwise influence the result, including instrumental errors related to the properties of the algorithms themselves or of the measurement devices. It is generally reasonable to state that the total error is the sum of partial errors [1]

$$e = e_1 + e_2 + \dots + e_J, \quad (2)$$

where J is the total number of errors.

One of the fundamental concepts used to describe the result of a measurement is the estimate of the measurand, defined as the random number closest to the true measurand. This means that the estimate is free from any systematic errors that are removed from the raw result during processing. Denoting the estimate by $\hat{x}$ and taking into account that the error definition (1) can be applied to different types of error, the formula for the estimate error is obtained as

$$\hat{e} = x - \hat{x}. \quad (3)$$

According to the definition in [4], the systematic error e_{sys} is the component of the error e of the raw result that can be determined and thus removed from it. This leads to the following expression for the estimate error

$$\hat{e} = e - e_{\text{sys}}. \quad (4)$$

Removing the systematic component from the error must be accompanied by the addition of a correction c to the raw measurement result $\tilde{x}$, thereby yielding an estimate of the measurand. Thus,

$$\hat{x} = \tilde{x} + c. \quad (5)$$

Combining expressions (3), (4), and (5), we obtain

$$e - e_{\text{sys}} = x - \hat{x} = x - (\tilde{x} + c) = e - c, \quad (6)$$

which means that the correction $c = e_{\text{sys}}$. Taking this into account, it follows from (5) that systematic error correction can be achieved by adding the systematic component to the raw measurement result.

Example 1. The raw result obtained from an A/D converter that measures the voltage x using a standard with a quantum structure is described by the relationship [1]

$$\tilde{x} = q \cdot \text{ent}\left(\frac{x}{q}\right), \quad (7)$$

in which the function ‘ent’ determines the integer part of its argument. Assuming that the values of the measurand are equally probable over the entire processing range, the quantization error e_q follows a uniform distribution over the interval from 0 to q . The error probability density function is then given by

$$g(e_q) = \begin{cases} \frac{1}{q} & \text{for } 0 \leq e_q \leq q \\ 0 & \text{for others} \end{cases}. \quad (8)$$

According to this definition, the systematic error is the systematic component of the error generally described as random. This means that each realization of this type of error contains the same systematic error value. This, in turn, leads to the conclusion that the basic component of the systematic error can be determined as the expected value of the random error distribution. For an error described by the probability density function (8), one obtains

$$e_{\text{sys}} = E[g(e_q)] = \int_0^q g(e_q) e_q de_q = \frac{1}{q} \cdot \frac{e_q^2}{2} \Big|_0^q = \frac{q}{2}. \quad (9)$$

It should be noted that the definition of systematic error given by equation (9) applies only to errors with distributions that are symmetric after removal of the expected value. The properties of the deductive method mean that the determination of the uncertainty interval, discussed in the next section, leads to the appearance of a second component of systematic error related to the asymmetry of the distribution.

Eliminating the systematic component e_{sys} involves subtracting it from each value of the raw-result error. This operation yields the following

probability density function for the estimate's quantization error

$$g(\hat{e}_q) = \begin{cases} \frac{1}{q} & \text{for } -\frac{q}{2} \leq \hat{e}_q \leq \frac{q}{2}, \\ 0 & \text{for others} \end{cases} \quad (10)$$

which means that after this subtraction, the resulting estimate error reflects only its random properties. In many cases, the distribution of the estimation error is symmetric, including the case described by expression (10). Consequently, the systematic error represented by the expected value does not contain a component associated with asymmetry in the distribution of the estimation error. Eliminating the systematic component e_{sys} from the error must be accompanied by correction of the systematic error in the raw result through the addition of a correction c equal to this component. Therefore, according to relations (4) and (9), the estimate of the measurand obtained as a result (7) of quantization is described by the expression

$$\hat{x} = \tilde{x} + c = q \cdot \text{ent}\left(\frac{x}{q}\right) + \frac{q}{2} = q \left[\text{ent}\left(\frac{x}{q}\right) + \frac{1}{2} \right]. \quad (11)$$

CORRECTION OF SYSTEMATIC ERROR AS A RESULT OF DETERMINING THE MEASURAND INTERVAL

Expressing the measurand as a numerical interval is the final step in describing the uncertainty of a measurement result. The basis for determining the measurand interval is the definition of the uncertainty interval, which is consistent with the definition of measurement error (1). Both of these definitions form the foundation of the deductive method.

The uncertainty interval is defined as a numerical interval within which the difference between the true value of the measurand and its measurement result lies with a given probability p . For the raw result, this definition is formally expressed as

$$P[\underline{u} \leq (x - \tilde{x}) \leq \bar{u}] = p, \quad (12)$$

where: P denotes probability, and $\underline{u}$, $\bar{u}$ denote the lower and upper limits of the uncertainty interval, respectively.

Definition (12) shows that the uncertainty interval is determined from the set of error values

of measurement results, because, in accordance with the definition of error (1), the expression (12) can be written in the form

$$P[\underline{u} \leq e \leq \bar{u}] = p, \quad (13)$$

which shows that the limits of the uncertainty interval can be determined as parameters of the error distribution. Using the error probability density function $g(e)$, the lower limit of the interval is calculated from the functional

$$\int_{-\infty}^{\underline{u}} g(e) de = \frac{1-p}{2}, \quad (14)$$

and the upper limit is calculated according to the expression

$$\int_{\bar{u}}^{\infty} g(e) de = \frac{1-p}{2}, \quad (15)$$

where p is the confidence level, usually set to $p = 0.95$ [5]. When using the Monte Carlo method, the error distribution is represented by a histogram. The boundaries of the uncertainty interval are then determined as equal-tail empirical quantiles in a manner corresponding to Equations 14 and 15. From the definition of the uncertainty interval (12), it follows that, with probability p , the following inequality is satisfied

$$\tilde{x} + \underline{u} \leq x \leq \tilde{x} + \bar{u}, \quad (16)$$

meaning that the true value of the measurand is within the range

$$\tilde{x} = \tilde{x} + [\underline{u}; \bar{u}] = [\tilde{x} + \underline{u}; \tilde{x} + \bar{u}], \quad (17)$$

called the measurand interval. It is an interval description of the raw measurement result that takes into account the random dispersion of the results that can be obtained under specific measurement conditions. According to the expression (17), the measurand interval is the sum of the uncertainty interval

$$\tilde{u} = [\underline{u}; \bar{u}], \quad (18)$$

and the raw measurement result $\tilde{x}$.

Similar reasoning can be applied when the estimate of the measurand, rather than the raw result, is used in definition (12) of the uncertainty interval. It then follows from this definition that, with probability p , the true value of the measurand lies in the interval

$$\hat{x} + \underline{u} \leq x \leq \hat{x} + \bar{u}, \quad (19)$$

where $\underline{\hat{u}}$ and $\widehat{\bar{u}}$ are the lower and upper limits of the uncertainty interval determined with respect to the estimate of the measurement result. Therefore, the measurand interval in this case has the form

$$\hat{x} = \hat{x} + [\underline{\hat{u}}; \widehat{\bar{u}}] = [\hat{x} + \underline{\hat{u}}; \hat{x} + \widehat{\bar{u}}], \quad (20)$$

and the uncertainty interval is described as

$$\widehat{\bar{u}} = [\underline{\hat{u}}; \widehat{\bar{u}}]. \quad (21)$$

Removing the systematic component reduces the limits of the uncertainty interval for the estimate error by the systematic component relative to the limits determined for the raw result error, which means that

$$\underline{\hat{u}} = \underline{u} - e_{\text{sys}}, \quad (22)$$

Taking into account relation (4) and the fact that the correction c is equal to the systematic component e_{sys} of the error, the measurand interval (20) takes the form

$$\hat{x} = [\hat{x} + \underline{\hat{u}}; \hat{x} + \widehat{\bar{u}}] = [\tilde{x} + c + \underline{u} - e_{\text{sys}};$$

$$\tilde{x} + c + \bar{u} - e_{\text{sys}}] = [\tilde{x} + \underline{u}; \tilde{x} + \bar{u}] = \tilde{x}, \quad (23)$$

which leads to the important conclusion that the measurand interval has the same form whether determined for the raw result or for the measurand estimate.

The uncertainty of the measurement result of the measurand is defined as the radius of the uncertainty interval, determined according to the rules of interval arithmetic as half the length of the interval [13], i.e., the average of the difference of its boundaries. Therefore, in accordance with (18) and (22), the uncertainty is described by the expression

$$u = \text{rad}(\widehat{\bar{u}}) = \frac{\bar{u} - \underline{u}}{2} = \frac{(\widehat{\bar{u}} + e_{\text{sys}}) - (\underline{\hat{u}} + e_{\text{sys}})}{2} = \frac{\widehat{\bar{u}} - \underline{\hat{u}}}{2}, \quad (24)$$

which means that the uncertainty u can be determined as the radius of the uncertainty interval for both the raw result error and the estimate error.

Uncertainty can also be calculated as the radius of the measurand interval, since

$$\text{rad}(\tilde{x}) = \frac{\tilde{x} + \bar{u} - (\tilde{x} + \underline{u})}{2} = \frac{\bar{u} - \underline{u}}{2} = u. \quad (25)$$

It follows from this relationship and from (24) that there are three possible ways of determining uncertainty that give the same results.

It should be added that in a significantly larger number of cases the error of the estimate is symmetric about the vertical axis, in which case the absolute values of the limits of the uncertainty interval are equal and have opposite signs; that is,

$$u = -\underline{\hat{u}} = \widehat{\bar{u}}. \quad (26)$$

Another important property of the uncertainty interval concerns the systematic error. According to [13], the middle of the interval is the arithmetic mean of its boundaries. Therefore, based on relation (22), the middle of the uncertainty interval determined for the raw measurement result is described by the expression

$$\begin{aligned} \text{mid}(\widehat{\bar{u}}) &= \frac{\bar{u} + \underline{u}}{2} = \\ &= \frac{(\widehat{\bar{u}} + e_{\text{sys}}) + (\underline{\hat{u}} + e_{\text{sys}})}{2} = \frac{\widehat{\bar{u}} + \underline{\hat{u}}}{2} + e_{\text{sys}}, \end{aligned} \quad (27)$$

meaning that the middles of the uncertainty intervals calculated for the raw measurement result and for its estimate differ by the value of the systematic error. In accordance with relation (26), for a symmetric error of the estimate, the mean value of the limits of the uncertainty interval is equal to zero. In such a case, expression (27) takes the form

$$\text{mid}(\widehat{\bar{u}}) = e_{\text{sys}}, \quad (28)$$

which means that the systematic error can then be calculated from the uncertainty interval determined for the raw measurement result. Therefore, there is no need to remove the systematic component from the partial error distributions, which is particularly important in complex measurement situations where multiple types of error are present. The above conclusion applies when the distribution of the estimate error is symmetric. If the error distribution is asymmetric after the systematic component has been removed, an additional term appears in expression (27). This component can be interpreted as the systematic error caused by the asymmetry of the random component of the estimate error distribution. From the point of view of metrological analysis, the measurand estimate and its uncertainty form a pair of numbers defining the measurand interval, which is widely used for the interval description of the measurand after its measurement [5]. The estimate can be obtained by applying a systematic-error correction to the raw measurement result, as shown in Example 1. However, another, much simpler method can be used to obtain the value of the estimate when the

estimate is affected by many errors. This method consists of determining the middle of the measurement interval, which, according to (23), is the same regardless of the method used to obtain it. For the measurand interval determined with respect to the estimate, we obtain

$$\text{mid}(\hat{x}) = \frac{\hat{x} + \hat{u} + \hat{x} + \hat{u}}{2} = \hat{x} + \frac{\hat{u} + \hat{u}}{2} = \hat{x}, \quad (29)$$

where we take into account that the sum of the limits of the symmetric uncertainty interval is 0. From expression (29), it follows that the estimate of the measurand can be calculated as the middle of the measurand interval determined for the raw measurement result. This allows the estimate to be obtained without applying systematic error corrections to the raw result when there are many partial errors with systematic components.

Example 2. The measurement result is obtained using a voltage A/D converter whose quantum is $q = 1$ mV. For the error in the raw quantization result described by the relationship (8), the lower limit of the uncertainty interval is calculated using functional (14), which yields the expression

$$\int_0^u g(e_q) de_q = \int_0^u \frac{1}{q} de_q = \frac{1 - 0.95}{2}, \quad (30)$$

from which it follows that, for the confidence level $p = 0.95$, this limit has the value

$$\begin{aligned} \underline{u} &= 0.025 q = 0.025 \cdot 1 \cdot 10^{-3} = \\ &= 0.025 \cdot 10^{-3} \text{ V}. \end{aligned} \quad (31)$$

According to formula (15), the upper bound is obtained by solving the functional

$$\int_{\underline{u}}^{\bar{u}} \frac{1}{q} de_q = \frac{1 - 0.95}{2}, \quad (32)$$

from which it follows that the upper limit of the uncertainty interval has the value

$$\begin{aligned} \bar{u} &= 0.975 q = 0.975 \cdot 1 \cdot 10^{-3} = \\ &= 0.975 \cdot 10^{-3} \text{ V}. \end{aligned} \quad (33)$$

The uncertainty interval determined according to (18) has the form

$$\vec{u} = [\underline{u}; \bar{u}] = [0.025; 0.975] \text{ mV}, \quad (34)$$

and the uncertainty calculated from relationship (25) as the radius of the uncertainty interval is

$$\begin{aligned} u &= \text{rad}(\vec{u}) = \frac{0.975 - 0.025}{2} \cdot 10^{-3} = \\ &= 0.475 \cdot 10^{-3} \text{ V}. \end{aligned} \quad (35)$$

The systematic error calculated according to (28) as the middle of the uncertainty interval (34) determined for the raw measurement result has the value

$$\begin{aligned} e_{\text{sys}} &= \text{mid}(\vec{u}) = \frac{0.975 + 0.025}{2} \cdot 10^{-3} \text{ V} = \\ &= 0.5 \text{ mV}, \end{aligned} \quad (36)$$

the same as the systematic component of the error calculated from expression (9) for $q = 1$ mV.

Example 3. The error with the distribution described by expression (8) has, according to (9), an expected value of $q/2 = 1/2 = 0.5$ mV, which is equal to the systematic error. After removing the systematic component, the error of the estimate is described by a symmetric probability density function (10). The lower bound of the uncertainty interval is calculated for this error according to the expression

$$\begin{aligned} \int_{-\infty}^{\underline{u}} g(\hat{e}_q) d\hat{e}_q &= \int_{-q/2}^{\underline{u}} \frac{1}{q} d\hat{e}_q = \\ &= \frac{1 - 0.95}{2}, \end{aligned} \quad (37)$$

which yields

$$\underline{u} = -0.475 \cdot 10^{-3} \text{ V}. \quad (38)$$

For the upper limit of the uncertainty interval, we have

$$\int_{\bar{u}}^{q/2} \frac{1}{q} d\hat{e}_q = \frac{1 - 0.95}{2}, \quad (39)$$

therefore, its value is

$$\bar{u} = 0.475 \cdot 10^{-3} \text{ V}. \quad (40)$$

In the case under consideration, the uncertainty interval is symmetric and has the form

$$\hat{\vec{u}} = [\hat{\underline{u}}; \hat{\bar{u}}] = [-0.475; 0.475] \text{ mV}. \quad (41)$$

The uncertainty calculated for the symmetric uncertainty interval (41) is

$$\begin{aligned} u &= \text{rad}(\hat{\vec{u}}) = \frac{0.475 - (-0.475)}{2} \cdot 10^{-3} = \\ &= 0.475 \cdot 10^{-3} \text{ V}. \end{aligned} \quad (42)$$

and is the same as that obtained from the uncertainty interval (35), determined for the error in the raw quantization result.

Example 4. The reading of a 12-bit A/D converter, expressed as the number of quanta assigned to the measured voltage, is $n_q = 3577$. For the quantum value $q = 1$ mV, the raw measurement result for this measurand is obtained as the product [1]

$$\tilde{x} = n_q q = 3577 \cdot 0.001 = 3.577 \text{ V.} \quad (43)$$

For this result, based on relations (17) and (34), the measurand interval has the form

$$\begin{aligned} \tilde{x} &= [3.577 + 0.025 \cdot 10^{-3}; 3.577 + \\ &+ 0.975 \cdot 10^{-3}] = [3.577025; 3.577975] \text{ V.} \end{aligned} \quad (44)$$

The estimate of the measurand obtained as the middle of the measurand interval (44) determined for the raw result (43) has the value

$$\hat{x} = \text{mid}(\tilde{x}) = \frac{3.577025 + 3.577975}{2} = 3.5775 \text{ V,} \quad (45)$$

and the uncertainty calculated as the radius of the interval (44) has the value

$$u = \text{rad}(\tilde{x}) = \frac{3.577975 - 3.577025}{2} = 0.475 \text{ mV.} \quad (46)$$

SYSTEMATIC ERROR CORRECTION IN THE PROCESSING OF MEASUREMENT RESULTS SERIES

The considerations presented in the previous section indicate that systematic error correction is performed ‘automatically’ by combining the raw result and the uncertainty interval into a measurand interval. The simplest way to obtain the value of the systematic error is to calculate the middle of the uncertainty interval. The effectiveness of these methods is particularly evident in algorithms that process measurement data sequences, in which multiple sources of error are present and the algorithm processes many measurement results. This leads to the conclusion that the analysis of systematic error is closely linked to the determination of uncertainty. This section presents general aspects of applying the deductive method to the metrological analysis of processing algorithms, with an emphasis on issues related to systematic errors. To simplify the discussion, the problem was analyzed using a simple linear algorithm that processes measurands affected by two types of error.

A linear algorithm is a linear combination of measurands with constant coefficients, which means that it can be written as

$$z = a_0 x_0 + a_1 x_1 + \dots + a_{K-1} x_{K-1}, \quad (47)$$

where: $x_0, \dots, x_{K-1}$ are input measurands, z is the output measurand, and $a_0, \dots, a_{K-1}$ are constant coefficients.

The basis for the metrological analysis of an algorithm is the measurand model obtained by transforming the error definition (1). For the raw result, this model takes the form

$$x = \tilde{x} + e, \quad (48)$$

according to which the true value of the measurand is the sum of two random numbers: the measurement result and its error. Repeated measurements of the same measurand under the same conditions yield different random numbers, but relation (48) means that the realizations of the quantities on its right-hand side have opposite signs and differ by a constant value of the measurand. Given that uncertainty is a parameter that characterizes the dispersion of measurement results for the measurand under given measurement conditions [5], relation (48) allows us to conclude that the uncertainty determined from a set of error values can be applied to the description of the measurand. This property of the measurand model (48) is crucial for the metrological analysis of processing algorithms, as it allows equations to be formulated that describe error propagation through both linear [3] and nonlinear [2] algorithms. From relation (48), it follows that it is reasonable to determine the uncertainty of the algorithm’s output measurand based on the distribution of the measurand error.

For the purpose of describing its metrological properties, the algorithm is assumed to process a sequence of input measurands into a single output measurand. Using the measurand model (48) allows the processing algorithm given by (47) to be written as

$$\begin{aligned} z &= \tilde{z} + e = a_0(\tilde{x}_0 + e_0) + a_1(\tilde{x}_1 + e_1) \\ &+ \dots + a_{K-1}(\tilde{x}_{K-1} + e_{K-1}). \end{aligned} \quad (49)$$

Separating the terms of equation (49) yields two relations. The first, called the processing equation, describes the numerical operations performed on the measurement results of the input measurands to obtain the output result and has the form

$$\tilde{z} = a_0 \tilde{x}_0 + a_1 \tilde{x}_1 + \dots + a_{K-1} \tilde{x}_{K-1}. \quad (50)$$

The second relationship is the algorithm error equation

$$e = a_0 e_0 + a_1 e_1 + \dots + a_{K-1} e_{K-1}, \quad (51)$$

which defines the relationship between the realization of the total error e and the realizations of the partial errors $e_0, e_1, \dots, e_{K-1}$ affecting the input results. This equation is primarily used to analyze the propagation of errors through the algorithm [2,3] and to compare their contributions to the total error in the output result.

Each partial error can be represented as: $e_k = \hat{e}_k + e_{k,\text{sys}}$, $k = 0, 1, \dots, K-1$, i.e., as the sum of the error realization $\hat{e}_k$ that describes the random error of the measurand estimate and the systematic component $e_{k,\text{sys}}$. In this case, the error Equation 51 can be written in the form

$$\hat{e}_{\text{out}} + e_{\text{sys,out}} = a_0(\hat{e}_0 + e_{0,\text{sys}}) + a_1(\hat{e}_1 + e_{1,\text{sys}}) + \dots + a_{K-1}(\hat{e}_{K-1} + e_{K-1,\text{sys}}), \quad (52)$$

where the symbol ‘out’ denotes quantities at the output of the algorithm.

The input measurands form the measurement window [3], which means that they are measured under the same conditions, so their errors are described in the same way. Taking this into account, the error Equation 52 can be separated into two expressions. The first describes the relationships between the realizations of the random errors of the input measurand estimates and the output measurand. This equation has the form

$$\hat{e}_{\text{out}} = a_0 \hat{e}_0 + a_1 \hat{e}_1 + \dots + a_{K-1} \hat{e}_{K-1}, \quad (53)$$

and describes the propagation of random errors with the same distributions and zero expected value from the input to the output of the algorithm.

The second equation describes the propagation of systematic errors. Given that all these errors have the same value e_{sys} for each measurand, this equation becomes

$$e_{\text{sys,out}} = a_0 e_{\text{sys}} + a_1 e_{\text{sys}} + \dots + a_{K-1} e_{\text{sys}} = (a_0 + a_1 + \dots + a_{K-1}) e_{\text{sys}}. \quad (54)$$

In measurement practice, the measurand is affected by many partial errors, which may contain systematic components. According to relation (2), the systematic error at the output of the algorithm is then the sum of partial errors propagating according to relation (54).

Example 5. The algorithm for determining the average of a series has the form given by

Equation 47, in which the coefficients have the same value, denoted by a . For $K = 100$ results, we have

$$a_0 = a_1 = \dots = a_{K-1} = a = \frac{1}{K} = \frac{1}{100} = 0.01. \quad (55)$$

According to Equation 54, the systematic error at the output of the algorithm is given by

$$e_{\text{sys,out}} = e_{\text{sys}} \sum_{k=0}^{99} a_k = e_{\text{sys}} \cdot 100 \cdot 0.01 = e_{\text{sys}}, \quad (56)$$

which means that the systematic error at the output of the algorithm is the same as that in each of the input errors, and therefore the systematic error propagates from the input to the output of the algorithm unchanged. For the systematic component of the quantization error in Example 1, equal to $e_{\text{sys}} = q/2$ according to (9), and for the quantum value $q = 1$ mV, the systematic error at the output of the algorithm has the value

$$e_{\text{sys,out}} = e_{\text{sys}} = \frac{q}{2} = \frac{1}{2} = 0.5 \text{ mV}. \quad (57)$$

When input measurands are affected by multiple errors, determining their systematic components can be complex, as it requires knowledge of the probability density functions of partial errors. Furthermore, calculating the systematic error at the algorithm’s output requires determining the systematic error propagation equations. Even greater problems arise when attempting to analytically determine the random error distributions at the algorithm’s output using the propagation Equation 53. For these reasons, probabilistic simulation using the Monte Carlo method [14], as recommended in the supplement [15], is commonly used to determine the total error distribution. When the deductive method is used, the uncertainty interval at the algorithm’s output is determined based on the total error distribution, the middle of which, according to (28), is equal to the systematic error. This error is corrected in the final result by combining the result with the uncertainty interval, thereby eliminating the need for other operations involving systematic errors. The following experiment shows how to obtain the total error distribution at the output of an algorithm using the Monte Carlo method, assuming that the input measurands are affected by quantization errors and errors caused by noise.

Experiment. Under the adopted assumptions, the following procedure was used in the Monte Carlo simulation.

- 1) Draw a sample of the input voltage (measurand) x from a uniform distribution over the range of a unipolar 12-bit A/D converter with a quantum value of $q = 1$ mV.
- 2) Determine the raw measurement result for the measurand x according to (7)

$$\tilde{x} = q \cdot \text{ent}\left(\frac{x + v_{\text{noi}}}{q}\right), \quad (58)$$

where v_{noi} is a realization of normally distributed noise with an expected value of 0 and a standard deviation σ_{noi} equal to the quantum value q .

- 3) Repeat the second step 100 times for the same value of the measurand x determined in step 1 and then determine the average of the series of 100 results using the averaging algorithm

$$\tilde{z} = 0.01 \sum_{k=0}^{99} \tilde{x}_k. \quad (59)$$

Determine the error of the mean as

$$e_{\text{out}} = x - \tilde{z}, \quad (60)$$

and add it to the set of error values. Repeat steps 1 to 4 100,000 times.

Present the resulting set of error values in the form of a histogram, determine the standard deviation and the expected value, and calculate the limits of the uncertainty interval according to the expressions (14) and (15).

In the experiment, the 0.5-mV systematic component is contained only in the quantization error affecting the input measurands. This means that the systematic error at the algorithm's output, determined analytically using equation (56), which describes the propagation of this error, is 0.5 mV. This value is also consistent with the expected value of the error distribution shown in Figure 1. According it, the uncertainty interval with experimentally determined limits has the form

$$\bar{u} = [\underline{u}; \bar{u}] = [0.297; 0.703] \text{ mV}. \quad (61)$$

The systematic error, calculated as the middle of the uncertainty interval (61), has the value

$$\begin{aligned} e_{\text{sys,out}} &= \text{mid}(\bar{u}) = \frac{\underline{u} + \bar{u}}{2} = \\ &= \frac{0.703 + 0.297}{2} = 0.5 \text{ mV}. \end{aligned} \quad (62)$$

Assume that the input measurands have the same true value $x = 3.577528$ V. The average value of the first 100 measurement results determined at the first step of the Experiment is

$$\tilde{z} = 3.57710 \text{ V}. \quad (63)$$

The measurand interval determined by combining this output result from the example algorithm (59) with the uncertainty interval (61) is of the form

$$\begin{aligned} \bar{z} &= [3.5771 + 0.30 \cdot 10^{-3}; 3.5771 + \\ &+ 0.70 \cdot 10^{-3}] = [3.5774; 3.5778] \text{ V}. \end{aligned} \quad (64)$$

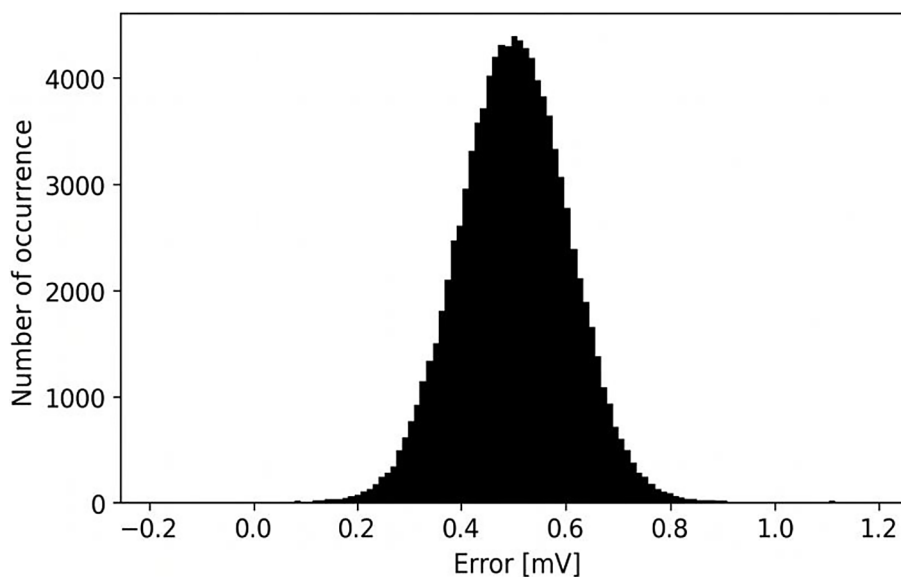


Figure 1. Histogram of the experimentally determined error. The standard deviation is $\sigma_{\text{out}} = 0.104$ mV, the expected value is 0.5003 mV, and the limits of the uncertainty interval are $\underline{u} = 0.297$ mV and $\bar{u} = 0.703$ mV

The middle of the measurand interval (64), according to (29), determines the value of the output measurand estimate, which is

$$\hat{z} = \text{mid}(\tilde{z}) = \frac{3.5774 + 3.5778}{2} = 3.5776 \text{ V.} \quad (65)$$

The measurement uncertainty of the output measurand at the confidence level $p = 0.95$ is, in accordance with expression (25), the radius of the uncertainty interval, which has the value

$$u = \text{rad}(\tilde{z}) = \frac{3.5778 - 3.5774}{2} = 0.2 \text{ mV.} \quad (66)$$

The same value can be obtained by determining the radius of the uncertainty interval (61):

$$u = \text{rad}(\tilde{u}) = \frac{0.703 - 0.297}{2} = 0.203 \cong 0.2 \text{ mV.} \quad (67)$$

According to the Central Limit Theorem [5], for identical or similar values of the coefficients of the algorithm processing measurands with the same errors, the distribution of the output error is close to normal. In such a case, the uncertainty at a confidence level of $p = 0.95$ can be calculated as twice the standard deviation [5]. Based on the value given in Figure 1, we obtain

$$u \cong 2\sigma_{\text{out}} = 2 \cdot 0.1 = 0.2 \text{ mV.} \quad (68)$$

From the relations (62), (65), and (66), it follows that all accuracy parameters characterizing the measurand at the output of the processing algorithm can be determined based on the distribution of its total error. The presented experiment shows that the Monte Carlo method is an effective tool for obtaining this distribution by simulation.

Simulation experiments require verification of the results to eliminate inaccuracies that may occur both when formulating the assumptions and when describing the measurement process. When the deductive method is used to define a formal framework for the experiment, error-propagation equations can be used to verify the results, thereby providing an analytical description of the error parameters determined by simulation at the algorithm's output. In particular, it is advisable to use the error equation to describe the propagation of random errors from the input to the output of the algorithm.

According to Equation 51, assuming no correlation between partial errors, the relationship between the standard deviation σ_{out} of the total error at the output of the algorithm under

consideration and the standard deviations of the input measurand errors is described by the expression

$$\begin{aligned} \sigma_{\text{out}} &= \sqrt{(a_0\sigma_0)^2 + (a_1\sigma_1)^2 + \dots + (a_{K-1}\sigma_{K-1})^2} \\ &= \sigma_{\text{in}} \sqrt{a_0^2 + a_1^2 + \dots + a_{K-1}^2}, \end{aligned} \quad (69)$$

where all input measurand errors have the same standard deviation σ_{in} . The coefficients of the example algorithm have the same value $a = 0.01$, for which the relationship (69) takes the form

$$\sigma_{\text{out}} = \sigma_{\text{in}} \sqrt{K a^2} = a \sigma_{\text{in}} \sqrt{K} = 0.01 \cdot \sqrt{100} \sigma_{\text{in}} = 0.1 \sigma_{\text{in}}. \quad (70)$$

For the purposes of the experiment, it was assumed that the input measurands are affected by two random errors: the quantization error and the error caused by normally distributed noise with a standard deviation $\sigma_{\text{noi}} = q = 1 \text{ mV}$. These errors are not correlated [1], so the standard deviation of their sum is given by

$$\sigma_{\text{in}} = \sqrt{\sigma_q^2 + \sigma_{\text{noi}}^2}, \quad (71)$$

where: σ_q is the standard deviation of the uniformly distributed quantization error (10), which is given by

$$\sigma_q = \frac{q}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.29 \text{ mV.} \quad (72)$$

According to relations (70), (71), and (72), the standard deviation of the error at the output of the algorithm is obtained as follows:

$$\sigma_{\text{out}} = 0.1 \sigma_{\text{in}} = 0.1 \sqrt{1^2 + 0.29^2} = 0.104 \text{ mV,} \quad (73)$$

which is consistent with the value obtained by simulation for the total error distribution shown as a histogram in Figure 1. This means that the results obtained by simulation can be considered reliable.

CLOSING REMARKS

This article proves that determining measurement uncertainty using the deductive method does not require correction of systematic errors in the raw results. This occurs because these errors are compensated when the uncertainty interval is combined with the raw measurement result. This property is particularly important for algorithms that process long measurement data sequences

affected by many systematic errors. The paper shows that the middle of the uncertainty interval is equal to the systematic error, which has two components. One is equal to the expected value of the error, and the other is related to the asymmetry of its distribution. Correcting this second component shifts the boundaries of the uncertainty interval relative to the boundaries determined after eliminating the expected value. It is debatable which uncertainty interval (with or without asymmetry error correction) better describes the inaccuracy of the measurement result, taking into account its determinants. The basis for determining uncertainty in the proposed manner is knowledge of the error distribution of the measurement result. The most effective way to obtain this distribution is to use the Monte Carlo method, recommended for measurement analysis [15]. In the deductive method, the distribution of the total error is obtained from a model of the measurement process, which represents the relationship between the output measurand and the input measurands, while accounting for the influence quantities. In the examples presented, such models are given by Equations 58 and 59. An important feature of the deductive method is that the error distribution obtained by simulation is verified using an analytical equation derived from the error-propagation equation. This equation makes it possible to determine the relationship between the standard deviations of the output and input errors. Consistency between the standard deviation obtained analytically and that calculated from the simulation result provides a reliable basis for stating that the uncertainty determined by the Monte Carlo method is correct. It should also be noted that the relationships obtained using the deductive method are relatively simple, especially when analyzing the properties of nonlinear algorithms for processing measurement data sequences. This is evident when the expressions used in [16] to obtain the uncertainty of the output results of the RMS algorithm using the GUM method are compared with those obtained in [2] for a similar active power algorithm using the deductive method. The deductive method described in this paper constitutes a coherent set of definitions and mathematical expressions used for the metrological analysis of the measurement process and the determination of the uncertainty of the results. After the relationships concerning systematic errors have been introduced, it is justified to claim that the method is complete in the

sense that it covers all measurement situations that can be described by a mathematical model representing the physical relationships between the output measurand, input measurands, and influence quantities. The procedures for determining uncertainty according to the deductive method can be used both in measurement laboratories and in industry. The first application is illustrated by the example describing the algorithm (59) for averaging the results of multiple measurements of the same measurand under the same conditions. In the monograph [1], the mathematical apparatus of the deductive method was used to analyze measurement processes using sensors with nonlinear static characteristics, operating in dynamic conditions and disturbed by many quantities typical of industrial conditions. It is planned to investigate the possibility of using empirical data obtained in the research [17] in the deductive method, which aims to determine the uncertainty of the results of the discrete wavelet transform. Moreover, it is planned to conduct a metrological analysis of the measurement process models described by implicit nonlinear models as considered in [18].

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