

Structural efficiency of beams with simple section variation laws under elemental and combined loads

Raúl Ccahuana Valdez¹, Fredy Bruno Nina Alarcón¹,
Jorge Luis Apaza Gutierrez^{1*}, David Franco Meneses Huanca¹

¹ Escuela Profesional de Ingeniería Mecánica, Facultad de Ingeniería de Producción y Servicios, Universidad Nacional de San Agustín de Arequipa, Arequipa 04000, Perú

* Corresponding author's e-mail: japazagut@unsa.edu.pe

ABSTRACT

This study investigates the structural response of lightweight non-prismatic beams with linear, quadratic, and trigonometric cross-sectional variation laws under elemental and combined loading conditions. The objective is to identify mechanically interpretable trends that support the preliminary selection of simple geometric parameterizations rather than to obtain an optimized geometry through iterative procedures. Beams restrained at one or both ends, comprising five boundary conditions and seven loading cases, were analyzed under an equal-volume criterion using a Euler-Bernoulli finite element formulation. The model was verified against benchmark solutions for non-prismatic beams, with a maximum relative error of 0.0301%, while the mesh-convergence study produced errors below 0.0034%. The configurations were compared using dimensionless coefficients based on maximum displacement, global displacement, and fundamental natural frequency. All reduced-section beams produced displacement coefficients greater than unity, indicating increased deformation relative to the uncut reference beam; consequently, the preferred parameterization is the one that minimizes this penalty. For beams restrained at one end, the trigonometric cut produced the lowest global displacement coefficient for every elemental and combined load, minimized the maximum displacement in most loading cases, and provided the highest frequency ratios. For beams restrained at both ends, the quadratic cut generally minimized both displacement coefficients and provided the highest frequency ratios for pinned-pinned and rotational spring-rotational spring supports. For fixed-fixed beams, the quadratic cut was generally preferable for displacement control, whereas the trigonometric cut better preserved the fundamental frequency. These trends are explained by the interaction between the spatial distribution of flexural rigidity, the bending-moment distribution, and the location of the removed mass. The results demonstrate that equal-volume section reductions can produce substantially different static and frequency-based responses, providing mechanics-based selection criteria for the preliminary design of lightweight non-prismatic beams.

Keywords: variable section beams, structural optimization, structural efficiency, Euler-Bernoulli, finite element method, support configurations, structural design.

INTRODUCTION

Lightweight structural components have become an essential requirement in modern mechanical engineering, where reducing structural mass contributes not only to material savings but also to improved energy efficiency, dynamic performance, payload capacity, and operational reliability. Consequently, engineers increasingly incorporate geometric weight-reduction strategies

during the preliminary stages of structural design, where multiple alternatives must be rapidly evaluated before detailed analyses are undertaken.

Among the various engineering applications, robotic manipulators provide a representative example because the reduction of moving mass directly enhances payload capacity while decreasing actuator effort and energy consumption. For instance, the upgrade of the EOD JVC 0.2 robotic arm increased its payload capacity

by approximately 66.7% through the incorporation of lightweight perforated structural members [1]. Similarly, the KVN inspection robot, developed for the confined air-intake ducts of Mirage 2000P aircraft, achieved significant reductions in structural weight and manufacturing complexity by employing lightweight curved structural components optimized through successive design and experimental iterations [2]. Despite their widespread use, geometric modifications can be implemented through numerous strategies, ranging from local perforations to continuous cross-sectional variations along the structural member. The selection of an appropriate geometric variation therefore becomes an important design decision, since different material distributions may significantly alter the stiffness, deformation, and overall structural performance of the component.

Among the different alternatives available, linear, polynomial, and trigonometric parameterizations have been widely employed owing to their balance between mathematical simplicity, numerical robustness, manufacturing feasibility, and geometric flexibility. Linear parameterizations have been extensively adopted in the analysis of non-prismatic and functionally graded beams because of their straightforward geometric definition and efficient numerical implementation [6]. Polynomial parameterizations provide additional flexibility for describing smoothly varying structural members while preserving a compact mathematical formulation. Among them, quadratic functions constitute the simplest nonlinear representation, introducing geometric curvature while maintaining a reduced number of design parameters and avoiding the unnecessary complexity associated with higher-order polynomials [4]. Likewise, trigonometric parameterizations have been successfully employed in structural optimization because they accurately represent smooth structural profiles using only a reduced number of design parameters [5]. Together, these three families constitute representative engineering-oriented geometric descriptions suitable for preliminary structural design. Despite their widespread use, however, their relative structural performance under different loading and support conditions has not been systematically compared to establish practical engineering guidelines for selecting an appropriate geometric parameterization.

Once the geometric parameterization has been defined, predicting the structural response

requires an appropriate mathematical formulation capable of accurately representing the influence of continuously varying cross sections. Consequently, considerable research efforts have been devoted to the development of analytical and numerical models for non-prismatic beams. Existing contributions include finite element formulations for coupled beam systems [6], polynomial representations of variable cross sections [7], Krylov–Duncan function expansions for modeling structural imperfections [8], Green-function-based finite element formulations for static and dynamic analyses [9], and, more recently, the Gaussian expansion element method (GEEM), which improves the numerical accuracy, convergence, and stability of dynamic analyses for highly non-uniform structures [10]. Furthermore, both Euler–Bernoulli and Timoshenko beam theories have been successfully employed to investigate the behavior of variable cross-section beams, depending on the relevance of shear deformation and rotary inertia [10]. Since the objective of the present work is to identify general structural trends associated with different geometric parameterizations during the preliminary design stage, rather than to capture local thick-beam effects, the Euler–Bernoulli formulation provides an appropriate balance between physical realism, mathematical simplicity, computational efficiency, and physical interpretability for slender beam applications [11].

In parallel with the development of increasingly accurate beam formulations, significant progress has also been achieved in structural optimization. Gradient-based approaches, such as the method of moving asymptotes (MMA) [12,13], together with metaheuristic techniques including particle swarm optimization (PSO) [14–15] and genetic algorithms (GA) [13], have demonstrated remarkable capability for identifying efficient structural configurations under prescribed objectives and constraints. Likewise, topology optimization methods, particularly the Solid Isotropic Material with Penalization (SIMP) approach, have become powerful tools for determining optimal material distributions within prescribed design domains [16]. These methodologies have considerably expanded the designer’s ability to obtain lightweight and high-performance structures.

Despite these advances, the resulting solutions are generally problem-specific because they depend on the selected objective function,

loading conditions, material properties, manufacturing constraints, and optimization strategy. Consequently, while optimization techniques are highly effective for obtaining optimal geometries for individual design problems, they do not necessarily provide general engineering criteria capable of guiding the selection of simple geometric parameterizations during the conceptual design stage, where rapid evaluation, physical interpretability, and straightforward implementation are often more valuable than obtaining a problem-dependent optimum. This distinction motivates the need for comparative studies capable of identifying general structural trends associated with representative geometric parameterizations, thereby providing practical design guidelines that complement, rather than replace, modern optimization techniques.

Although considerable progress has been achieved in the numerical analysis and optimization of variable cross-section beams, existing studies primarily focus on improving predictive models or obtaining optimal solutions for specific structural configurations. Consequently, comparatively little attention has been devoted to identifying general structural trends associated with representative geometric parameterizations under common loading and support conditions. As a result, designers still lack practical engineering guidelines capable of assisting the selection of simple cross-sectional variation laws during the conceptual design stage, where rapid evaluation and physical interpretability often take precedence over problem-specific optimization.

The present work systematically compares the structural response of beams with linear, quadratic, and trigonometric cross-sectional parameterizations under representative loading combinations and boundary conditions within a unified Euler–Bernoulli finite element framework. Instead of seeking an optimal geometry through iterative optimization procedures, the proposed approach aims to identify general structural trends capable of supporting engineering decisions during the preliminary design stage. To this end, the influence of the geometric parameterization is evaluated through dimensionless structural efficiency indices based on maximum displacement, global displacement, and natural frequency, providing practical design guidelines for selecting appropriate cross-sectional variation laws in lightweight structural applications.

METHODOLOGY

Problem definition

To investigate the influence of representative cross-sectional parameterizations on the structural response of lightweight beams, a two-dimensional non-prismatic beam subjected to representative combinations of concentrated loads, distributed loads, and bending moments is considered. The structural model, illustrated in Figure 1, consists of a beam of length L , constant width b , and continuously varying height $h(x)$, whose geometry is prescribed according to the selected parameterization. The beam is assumed to be homogeneous, isotropic, and linearly elastic, with material behavior characterized by Young's modulus E and density ρ . Different support configurations are subsequently considered to evaluate the influence of the geometric parameterization under representative structural design scenarios.

To investigate the influence of different geometric distributions under representative engineering scenarios, two families of beam configurations are considered, as illustrated in Figure 1.a Case I corresponds to beams with a single continuous cross-sectional reduction extending along the entire beam length, whereas Case II represents beams composed of two consecutive segments with independent geometric variations. For each family, linear, quadratic, and trigonometric parameterizations are investigated while preserving the same total structural volume.

Variation in height of the cross section

To ensure a consistent comparison among different geometric descriptions, all beam configurations share the same length, material properties, and cross-sectional dimensions at the clamped end. Furthermore, all parameterizations are constrained to have the same total structural volume, thereby ensuring identical structural mass for every beam analyzed. To satisfy this condition, the, denoted by the parameter (α), is independently determined for each minimum beam height geometric family.

Different values of (α) are obtained for the linear (α_1), quadratic (α_2), and trigonometric (α_3) parameterizations, such that all configurations contain the same amount of material despite their different geometric distributions. Under this normalization, the only variable considered

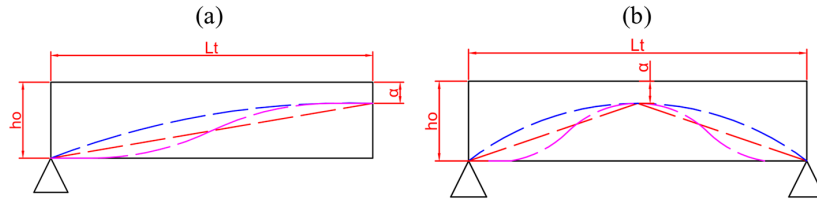


Figure 1. Beam analyzed with a linear cut (red line), quadratic cut (blue line) and trigonometric cut (pink line) for (a) Case I and (b) Case II

throughout the study is the mathematical law governing the longitudinal variation of the beam height. Three representative geometric parameterizations are therefore investigated: linear, quadratic, and trigonometric. These parameterizations were selected because they represent three widely adopted families of engineering-oriented geometric descriptions while preserving mathematical simplicity, numerical robustness, and manufacturing feasibility. Considering that the x -axis is horizontal and positive to the right with the point $x = 0$ located at the far left in both cases, for Case I, the linear law is defined as,

$$h_L(x) = \left(\frac{\alpha_1 - h_o}{L_t} \right) x + h_o \quad (1)$$

the quadratic law is defined by,

$$h_q(x) = \frac{h_o - \alpha_2}{L_t^2} (x - L_t)^2 + \alpha_2 \quad (2)$$

finally, the trigonometric law is expressed as,

$$h_{tri}(x) = (h_o - \alpha_3) \cos\left(\frac{\pi x}{2L_t}\right) + \alpha_3 \quad (3)$$

For Case II, the linear law is defined as:

the quadratic law is defined by:

$$h_q(x) = \frac{4(h_o - \alpha_2)}{L_t^2} \left(x - \frac{L_t}{2} \right)^2 + \alpha_2 \quad (5)$$

the trigonometric law is defined by:

$$h_{tri}(x) = \frac{h_o - \alpha_3}{2} \cos\left(\frac{2\pi x}{L_t}\right) + \frac{h_o + \alpha_3}{2} \quad (6)$$

To ensure a fair comparison among the different geometric parameterizations, an equal-volume criterion is adopted throughout this work. Instead of prescribing a common minimum cross-sectional height, the reduction parameter associated with each parameterization is selected such that all beam configurations retain the same percentage of the original structural volume. The relationship between the retained structural volume and the corresponding normalized minimum height (α/h_o) is presented in Figure 2 for both geometric families. For the comparative analyses presented in this work, a retained structural volume of 75% is adopted, resulting in ($\alpha_1 = 0.500h_o$, $\alpha_2 = 0.625h_o$, and $\alpha_3 = 0.312h_o$ for Case I, and $\alpha_1 = 0.500h_o$, $\alpha_2 = 0.625h_o$, and $\alpha_3 = 0.500h_o$ for Case

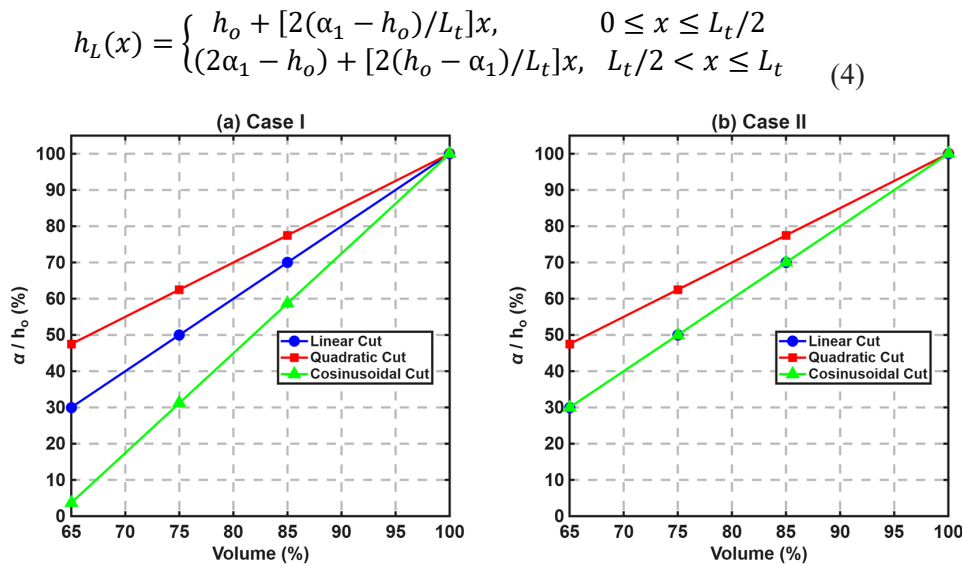


Figure 2. Volume-preserving normalization curves

II. Consequently, all beam configurations possess identical structural volume, ensuring that any differences observed in the structural response are exclusively associated with the adopted geometric parameterization.

MECHANICAL MODEL

The formulation is based on the classical Euler–Bernoulli beam theory under the assumptions of small deflections, linear elastic material behavior, homogeneous and isotropic material properties, and slender beam geometry, such that cross sections remain plane and perpendicular to the neutral axis after deformation. Consequently, shear deformation and rotary inertia are neglected, making the formulation suitable for slender structural members whose response is predominantly governed by bending. Under these assumptions, the equation governing the transverse motion of a non-prismatic beam subjected to axial compression, distributed loading, concentrated transverse forces, and externally applied bending moments can be written as,

$$\rho A(x) \frac{\partial^2 W}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 W}{\partial x^2} \right) - P \frac{\partial^2 W}{\partial x^2} = P_{tr} + q(x) + M_o \quad (7)$$

where: ($W(x, t)$) denotes the transverse displacement, (ρ) is the material density, ($A(x)$), is the cross-sectional area, ($I(x)$) is the second moment of area, (E) is Young's modulus, (P) is the applied axial compressive load, ($q(x)$) represents the distributed transverse load, (P_{tr}) denotes the concentrated transverse load, and (M_o) is the externally applied bending moment.

Since the beam width remains constant throughout the study, the cross-sectional properties are expressed as ($A(x) = b h(x)$ and $I(x) = b h^3(x)/12$), where (b) is the constant beam width and ($h(x)$) is the height distribution defined by the selected geometric parameterization.

The governing equation is solved under five representative support configurations and seven loading cases commonly encountered in structural engineering applications. The support configurations include fixed-free (Fi–Fr), rotational spring-free (RS–Fr), pinned-pinned (Pi–Pi), fixed-fixed (Fi–Fi), and rotational spring-rotational spring

(RS–RS) beams. The loading cases comprise concentrated transverse loads, uniformly distributed loads, externally applied bending moments, and their corresponding combinations. Together, these configurations provide a representative set of structural scenarios for evaluating the influence of the selected geometric parameterizations.

FINITE ELEMENT FORMULATION

The governing equation is discretized using the finite element method (FEM). The transverse displacement field ($W(x, t)$) is approximated using the classical cubic Hermitian interpolation functions, which satisfy the continuity requirements of the Euler–Bernoulli beam theory by ensuring continuity of both the transverse displacement and its first derivative between adjacent elements. Accordingly, the displacement field within each finite element is expressed as,

$$W(x, t) = [N(x_o)] \{d(t)\} \quad (8)$$

where: a interpolation matrix is,

$$[N(x_o)] = \begin{bmatrix} 1 - 3\left(\frac{x_o}{L}\right)^2 + 2\left(\frac{x_o}{L}\right)^3, \\ x_o - 2\frac{x_o^2}{L} + \frac{x_o^3}{L^2}, \\ 3\left(\frac{x_o}{L}\right)^2 - 2\left(\frac{x_o}{L}\right)^3, \\ -\frac{x_o^2}{L} + \frac{x_o^3}{L^2} \end{bmatrix} \quad (9)$$

where: (x_o) is the position from the local reference system of each finite element, located at the left node of each finite element and L is the length of this finite element.

Substituting the Hermitian interpolation, Equation 9, into the governing equation, Equation 7, and applying the standard finite element formulation yields the elemental equilibrium equation. To preserve a compact mathematical presentation, the resulting expressions are maintained in terms of the Hermitian shape function matrix and its derivatives, since their explicit polynomial expansion considerably increases the length of the formulation without providing additional physical insight. The elemental equation can therefore be expressed in Equation 10. It is worth noting that, in Equation 10, both the bending stiffness and inertia contributions

$$\sum_{i=1}^n \left(E \int_0^L I(x_o) [N'']^T [N''] \{d\} dx_o + P \int_0^L [N']^T [N'] \{d\} dx_o + \rho \int_0^L A(x_o) [N]^T [N] \{\ddot{d}\} dx_o \right) = \sum_{i=1}^n \int_0^L [N]^T \{Q_i\} dx_o \quad (10)$$

where: (n) is the total number of finite elements employed in the discretization, $\{Q_i\}$ is the elemental nodal load vector associated with each loading case, and $\{d\}$ and $\{\ddot{d}\}$ denote the elemental nodal displacement and acceleration vectors, respectively.

depend explicitly on the position through the cross-sectional properties ($I(x_o)$) and ($A(x_o)$). Consequently, unlike prismatic beam formulations, the elemental matrices vary from one finite element to another according to the local geometric parameterization, thereby allowing the finite element model to accurately represent the continuously varying cross section along the beam length.

Consequently, unlike conventional prismatic beam formulations, the elemental matrices are no longer identical throughout the mesh but are individually evaluated according to the local cross-sectional geometry, allowing the continuously varying structural profile to be accurately represented within the finite element framework.

Numerical verification

Verification of the finite element formulation

Before applying the proposed formulation to the beam configurations investigated in this work, the numerical implementation was verified against benchmark solutions reported in the literature. The selected benchmark corresponds to the free vibration analysis of non-prismatic Euler–Bernoulli beams with elliptical cross sections subjected to an axial compressive load, as reported by [17]. This benchmark was selected because it involves continuously varying cross-sectional properties, making it particularly suitable for assessing the accuracy of the proposed finite element implementation. Figure 3 illustrates

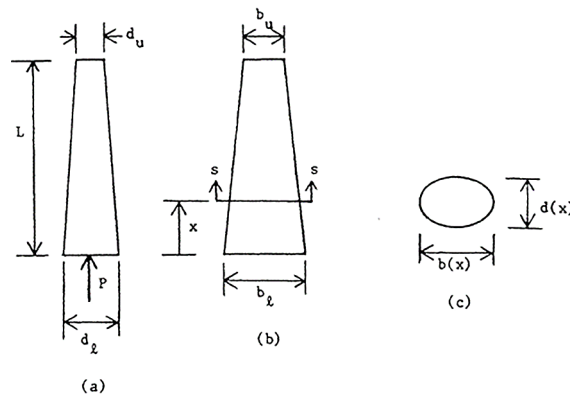


Figure 3. Tapered column under axial thrust: (a) linear depth taper; (b) linear width taper; (c) column cross-section at section s-s

Table 1. Validation with a dimensionless fundamental frequency

Fixed-Pinned connection with									
$\beta_c = 0.25$								$\beta_c = 2.00$	
α_c	[17]	Model	Relative error (%)	[17]	Model	Relative error (%)	[17]	Model	Relative error (%)
0.25	7.965	7.9643	0.0093	11.75	11.7483	0.0149	15.59	15.5889	0.0069
0.5	7.712	7.7115	0.0065	11.49	11.4866	0.0298	15.25	15.2464	0.0239
0.75	7.506	7.5059	0.0017	11.26	11.2566	0.0301	14.95	14.9482	0.0123
1	7.339	7.3389	0.0008	11.06	11.0613	0.012	14.69	14.6943	0.0291
1.5	7.083	7.0835	0.0075	10.75	10.7502	0.0023	14.29	14.2874	0.0181
2	6.894	6.8949	0.0124	10.51	10.5123	0.0221	13.97	13.9741	0.029
4	6.446	6.4452	0.0122	9.923	9.923	0.0005	13.19	13.1908	0.0058

Table 2. Base parameters of the beam

Parameter	Symbol	Value	Units
Length	L_t	1	m
With	b_o	0.1	m
Height	h_o	0.1	m
Young's modulus	E	160×10^9	N/m ²
Density	ρ	2300	Kg/m ³

the three benchmark configurations together with the corresponding cross-sectional variation functions adopted by Coulter [17], The geometry of the column has the linearly varying width and depth of the elliptical cross-section are defined as,

$$b(x) = b_l[1 + (\alpha_c - 1)x/L] \quad (11)$$

$$d(x) = d_l[1 + (\beta_c - 1)x/L] \quad (12)$$

where: $\alpha_c = b_u/b_l$ and $\beta_c = d_u/d_l$ being the taper ratios, and the subscripts u and l denoting the upper and lower ends, respectively.

Table 1 compares the dimensionless frequencies $\omega \sqrt{\rho A_l L_t^4 / EI_l}$, obtained from the reference and the present model. The results confirm the accuracy of the proposed formulation, with a maximum relative error of 0.0301%, demonstrating its high precision.

Mesh convergence study

After verifying the finite element implementation against an independent benchmark, a mesh convergence analysis was performed to determine

the minimum discretization required for the subsequent parametric study. A prismatic Euler–Bernoulli beam was selected for this purpose because analytical solutions are available, allowing the numerical error to be evaluated independently of the cross-sectional parameterizations investigated in the present work.

The beam geometry, material properties, boundary conditions, and loading configuration adopted for the convergence analysis are identical to those employed in the analytical solution, see Table 2. Five axial contraction load levels were considered, corresponding to ($\zeta = P/P_{cr} = 0$), (0.2), (0.4), (0.6), and (0.8), in order to verify that the selected mesh remains accurate throughout the entire range of compressive loads analyzed. In this study, the number of elements was varied and the relative error (%) in the first natural frequency was evaluated against the analytical solution for a pinned-pinned beam of uniform cross-section subjected to axial load presented in Rao (2018):

$$\omega_o = \frac{\pi^2}{L_t^2} \left(\frac{EI_o}{\rho A_o} \right)^2 (n_1^4 - n_1^2 \zeta) \quad (13)$$

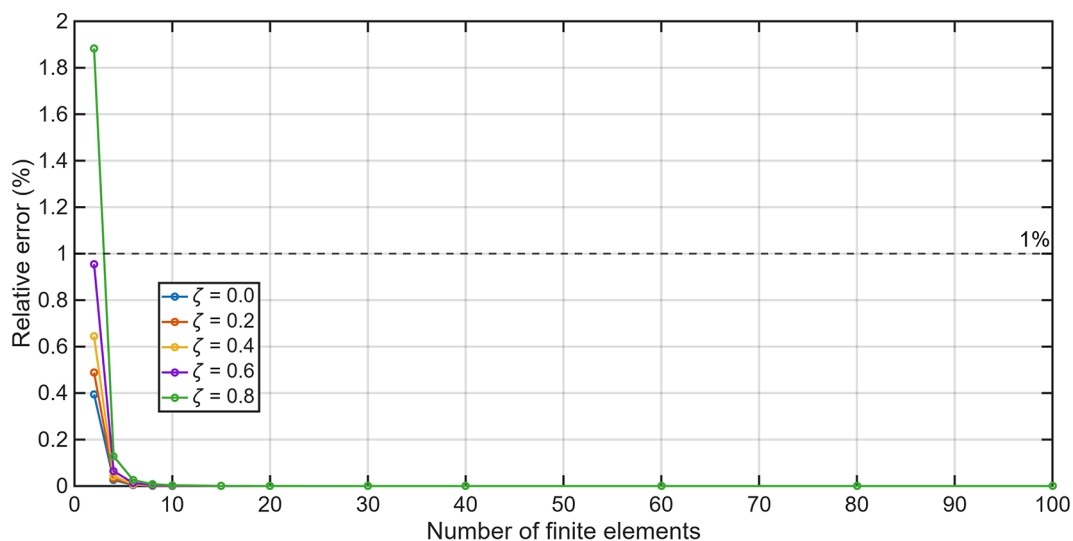


Figure 4. Convergence study for different axial loads

where: $n_1 = 1$ and is a mode of vibration, and the subscripts o denoting the point $x = 0$.

For each loading level, the beam was discretized using progressively finer meshes, and the corresponding relative error between the numerical and analytical natural frequencies was evaluated. As observed in Figure 4, all loading cases exhibit a similar convergence behavior despite the different axial load levels considered. Although coarse discretization produces larger discrepancies with respect to the analytical solution, the numerical error decreases rapidly as the mesh is refined. A discretization composed of ten Euler–Bernoulli beam elements provides a relative error below 0.0034% for all loading cases investigated and was therefore adopted throughout the remainder of this work. This mesh provides an excellent compromise between numerical accuracy and computational efficiency while ensuring that the subsequent comparisons among the different geometric parameterizations are not affected by discretization errors.

Determination of the representative rotational stiffness

To represent an intermediate support condition between the classical pinned and fixed boundaries, a representative rotational spring stiffness was determined through a parametric analysis. A cantilever Euler–Bernoulli beam of length L_t , constant rectangular cross section ($A_0 = bh_0$), material

density (ρ), and Young’s modulus (E) was considered, see Table 2. A uniformly distributed transverse load ($q_x = 10(b h_0)Ltg$), where $g = 9.8 \text{ m/s}^2$ was applied together with a compressive axial load equal to ($P = 0.2 P_{cr}$), ensuring that the beam remained within the linear prebuckling regime.

$$\eta(k_{rot}^{rep}) = \frac{W_{max}(k_{rot}) - W_{max}(\infty)}{W_{max}(0) - W_{max}(\infty)} \quad (14)$$

where: ($W_{max}(0)$) and ($W_{max}(\infty)$) denote the maximum transverse displacements of the pinned and fully fixed boundary conditions, respectively. Consequently, $\eta = 0$, corresponds to a fixed support, whereas ($\eta = 1$) represents a pinned support.

The representative rotational stiffness was selected by imposing ($\eta(k_{rot}^{rep}) = 0.5$), corresponding to the midpoint between both limiting boundary conditions. The variation of the normalized maximum transverse displacement with the rotational stiffness is presented in Figure 5.

According to this criterion, the representative rotational stiffness is ($\eta(k_{rot}^{rep}) = 1.98$). This value was subsequently adopted in all analyses involving rotational spring boundary condition,

$$\begin{aligned} EI(x) \frac{\partial^2 W}{\partial x^2} \bigg|_{\substack{\text{Case I: } x=0 \\ \text{Case II: } x=L_t \text{ and } x=0}} &= \\ &= k_{rot}^{rep} \frac{\partial W}{\partial x} \bigg|_{\substack{\text{Case I: } x=0 \\ \text{Case II: } x=L_t \text{ and } x=0}} \quad (15) \end{aligned}$$

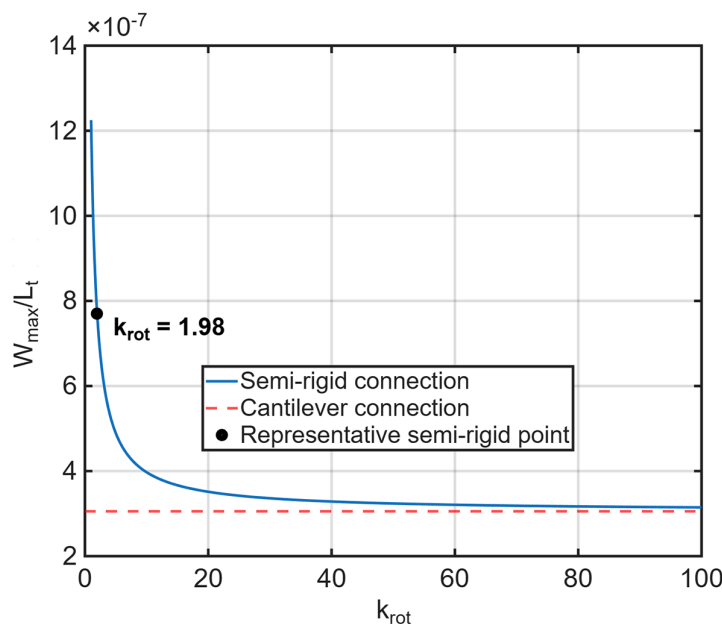


Figure 5. Maximum normalized transverse displacement W_{max}/L_t as a function of k_{rot}

Table 3. Loads

Load	Symbol	Value	Unit
Concentrated transverse load	P_{tr}	λM	[N]
Axial load	P	ζP_{cr}	[N]
Uniformly distributed	$q(x)$	$\lambda M/L_t$	[N/m]
Bending moment	M_o	$\lambda M L_t^2$	[N m]

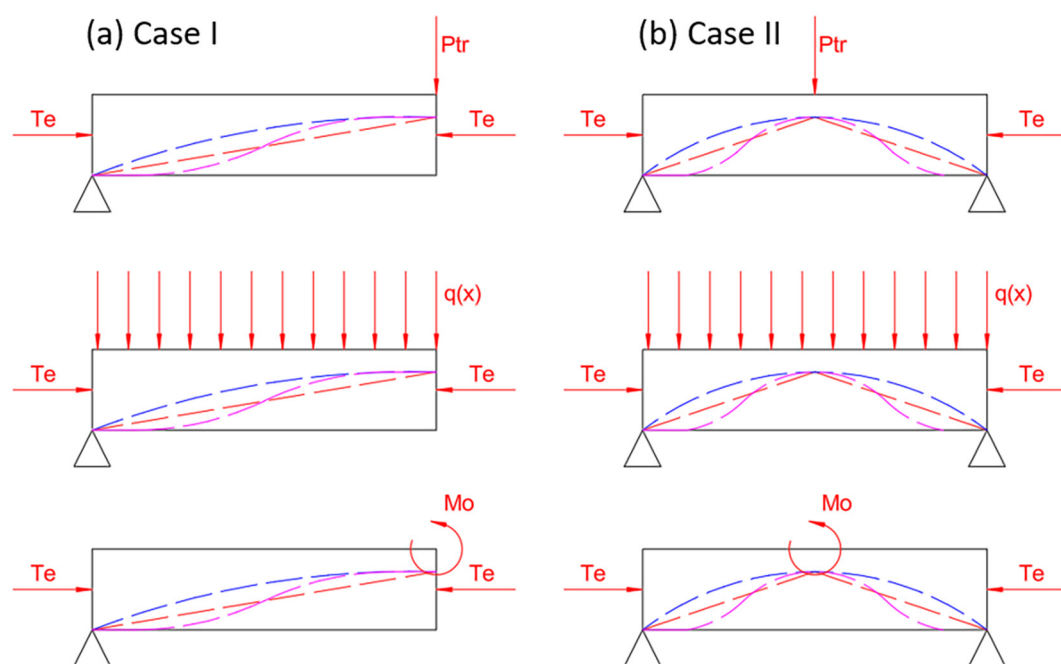
LOADING CONDITIONS

To ensure a consistent comparison among all beam configurations, the loading conditions are defined with respect to a common reference beam corresponding to the prismatic beam without cross-sectional reduction. The reference load is taken as the total self-weight, (M) of this beam and is subsequently multiplied by a dimensionless load coefficient, (λ). In the present study, a value of ($\lambda = 10$) is adopted, providing a loading level representative of practical engineering applications while maintaining the structural response within the linear elastic regime.

The resulting reference load is then used to define all external loading conditions considered throughout the study, including concentrated transverse loads, uniformly distributed loads, externally applied bending moments,

and their corresponding combinations. Consequently, every beam configuration, regardless of its geometric parameterization, support condition, or remaining structural volume, is evaluated under the same reference loading level. This procedure ensures that the observed differences in structural response arise exclusively from the adopted geometric parameterization rather than from variations in the applied external loads (Table 3).

Considering that P_{tr} and M_o is located at the free end for the case of one movement restriction, and at the midpoint for the case of two movement restrictions. The axial force P acts as a constant compressive axial load along the entire element and is incorporated into the formulation through the geometric stiffness matrix. Finally, the distributed transverse load $q(x)$ acts along the entire length of the element (Figure 6).


Figure 6. Elemental loads (a) Case I and (b) Case II

DIMENSIONLESS CHARACTERIZATION OF CROSS-SECTION REDUCTION EFFECTS

The results for the beams with reduced cross-sections were compared with those for the reference beam using three dimensionless coefficients,

$$\{\Phi_W\} = \frac{W_c}{W_o} \quad (16)$$

$$\{\Phi_{WG}\} = \frac{S_c}{S_o} \quad (17)$$

$$\{\Phi_\omega\} = \frac{\omega_c}{\omega_o} \quad (18)$$

where: W_c , S_c , and ω_c correspond, respectively, to the maximum displacement, the area swept by the deformed beam, and

the fundamental natural frequency of the beam with a reduced cross-section, while W_o , S_o , and ω_o represent the values obtained for the beam with a constant cross-section.

Therefore, for the displacement-based coefficients (Φ_W) and (Φ_{WG}), lower values indicate a more favorable structural response, since they correspond to smaller displacement penalties with respect to the reference beam. In contrast, for the frequency coefficient (Φ_ω), higher values are more favorable, since they indicate better preservation or improvement of the fundamental natural frequency. These coefficients allow for a dimensionless quantification of the effect of cross-section reduction on structural behavior, facilitating comparisons between different geometries, support conditions, and load types.

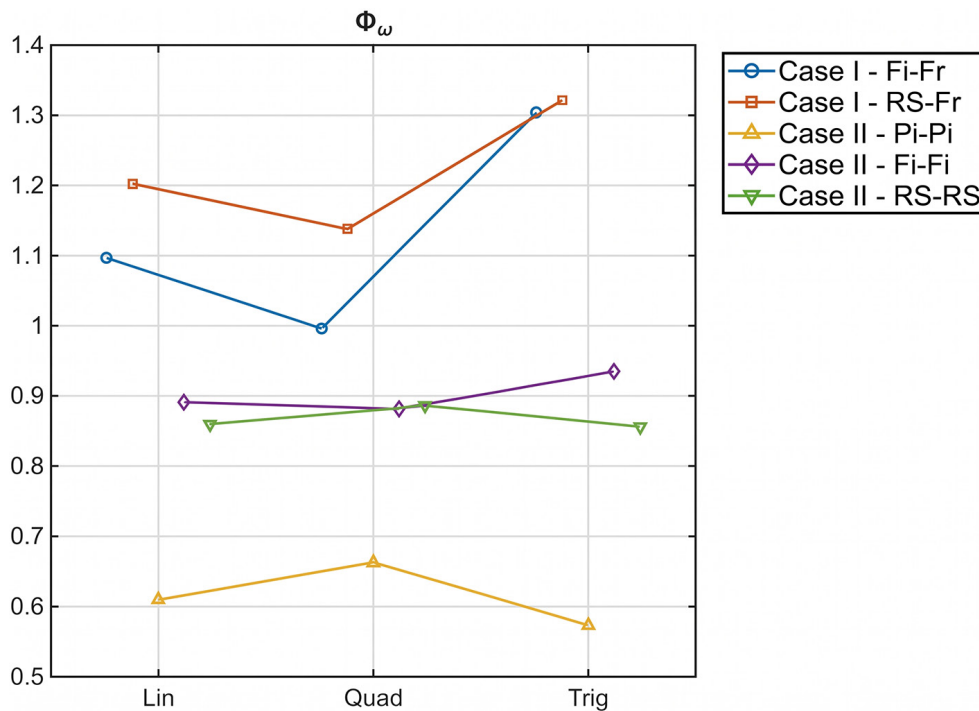


Figure 7. Fundamental frequency coefficient (Φ_ω) for different boundary condition configurations

Table 4. Fundamental frequency ratios and preferred parameterization according to the frequency-based criterion

Boundary condition		Linear	Quadratic	Trigonometric	Preferred frequency-based alternative
Case I	Fi-Fr	1.0966	0.9959	1.3041	Trigonometric
	RS-Fr	1.2025	1.1378	1.3210	Trigonometric
Case II	Fi-Fi	0.8910	0.8816	0.9348	Trigonometric
	Pi-Pi	0.6094	0.6629	0.5735	Quadratic
	RS-RS	0.8597	0.8861	0.8563	Quadratic

RESULTS

The obtained results allow analyzing the influence of load type, support configuration, and profile geometry on the coefficients based on displacement, Φ_w , on the global response measure, Φ_{WG} , and a coefficient based on frequency Φ_ω . For clarity, the results are grouped into elemental and combined load cases.

Fundamental frequency ratio

The influence of the cross-sectional parameterization on the dynamic response was evaluated through the fundamental frequency coefficient, $\Phi_\omega = \omega_c/\omega_o$, where (ω_c) is the fundamental natural frequency of the beam with reduced cross-section and ω_o is the corresponding value for the uncut reference beam. Therefore, values of $\Phi_\omega > 1$ indicate an increase in the fundamental frequency with respect to the reference configuration, whereas values lower than unity indicate a reduction.

Figure 7 and Table 4 reveal a consistent frequency-based ranking for the two Case I boundary conditions. The trigonometric cut produces the highest frequency ratios for both Fi–Fr and RS–Fr, with $\Phi_\omega = 1.3041$ and $\Phi_\omega = 1.3210$, respectively. It is followed by the linear cut, with corresponding values of $(\Phi_\omega = 1.0966)$ and $(\Phi_\omega = 1.2025)$, and finally by the quadratic cut, with values of $(\Phi_\omega = 0.9959)$ and $(\Phi_\omega = 1.1378)$. Thus, although the boundary condition affects the magnitude of the frequency ratio, it does not change the ranking among the three parameterizations. Relative to the uncut reference beam, the trigonometric and linear cuts increase the fundamental frequency for both support conditions, whereas the quadratic cut produces a slight reduction of 0.41% for Fi–Fr but an increase of 13.78% for RS–Fr.

For Case II, all frequency ratios remain below unity, indicating that the reduced-section configurations decrease the fundamental frequency relative to the uncut reference beam. However, the preferred parameterization depends on the boundary condition. For Fi–Fi, the trigonometric cut produces the highest ratio, $(\Phi_\omega = 0.9348)$, followed by the linear and quadratic cuts, with values of $(\Phi_\omega = 0.8910)$ and $(\Phi_\omega = 0.8816)$, respectively. In contrast, Pi–Pi and RS–RS exhibit the same general ranking: the quadratic cut gives the highest ratios, with values of $(\Phi_\omega = 0.6629)$ and $(\Phi_\omega = 0.8861)$, followed by the linear cut, with $(\Phi_\omega =$

0.6094) and $(\Phi_\omega = 0.8597)$, and the trigonometric cut, with $(\Phi_\omega = 0.5735)$ and $(\Phi_\omega = 0.8563)$. Therefore, unlike Case I, the frequency-based ranking in Case II changes with the support condition: the trigonometric parameterization is preferable for Fi–Fi, whereas the quadratic parameterization is preferable for Pi–Pi and RS–RS.

From a mechanical perspective, (Φ_ω) reflects the balance between the reduction in structural mass and the accompanying loss of flexural stiffness. Values greater than unity indicate that the effect of mass reduction compensates for, or exceeds, the stiffness loss, whereas values below unity indicate that the stiffness reduction has the dominant effect. Since all three cut parameterizations retain the same structural volume, the differences among their frequency ratios arise primarily from the spatial distribution of the retained material relative to the boundary conditions. For Case I, the trigonometric cut preserves more material near the constrained end and concentrates a greater section reduction toward the free end, producing the most favorable balance between retained flexural rigidity and reduced mass. For Case II, the reduction is concentrated within the span, making its influence more dependent on the support condition. The trigonometric cut provides the most favorable balance for Fi–Fi, whereas the greater central rigidity retained by the quadratic cut is favorable for Pi–Pi and RS–RS.

Overall, the frequency analysis indicates that the most suitable parameterization according to the frequency-based criterion does not necessarily coincide with the most favorable alternative according to the displacement-based criteria. For Case I, the trigonometric cut provides the highest frequency ratio for both support conditions. For Case II, the trigonometric cut is preferable for Fi–Fi beams, whereas the quadratic cut provides the highest frequency ratios for Pi–Pi and RS–RS configurations. Nevertheless, these frequency-based advantages must be considered together with Φ_w and Φ_{WG} before establishing a final design recommendation, since a parameterization that better preserves the fundamental frequency may still produce comparatively large static displacement penalties under certain loading conditions.

Elemental loads

Three elementary actions are considered: concentrated transverse load P , uniformly distributed load D , and externally applied bending moment M_o .

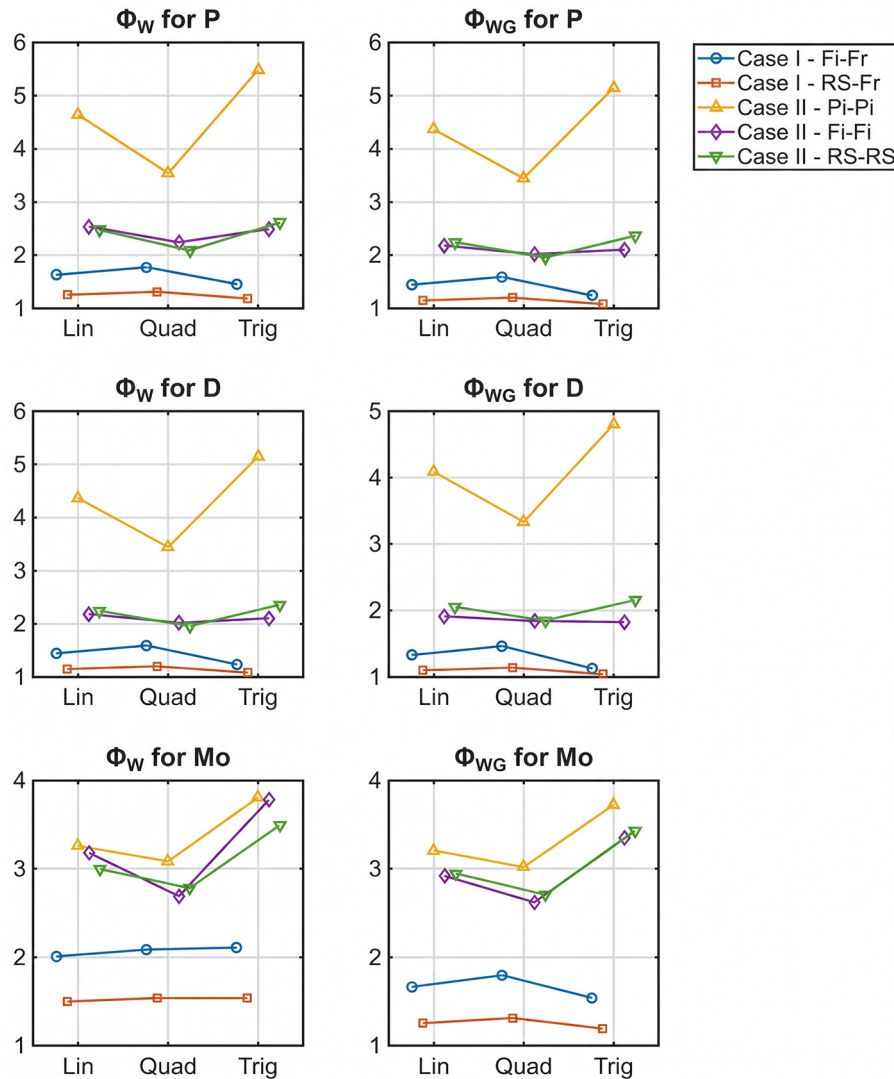


Figure 8. Displacement-based efficiency indices (Φ_W) and (Φ_{WG}) for the elemental load cases

The comparison is performed using the normalized indices (Φ_W) and (Φ_{WG}), which quantify, respectively, the variation of the maximum transverse displacement and the global displacement response with respect to the corresponding uncut reference beam. Since lower values of these coefficients indicate a smaller displacement penalty, the most favorable cut is the one that keeps both indices closer to unity. The elemental loading results are presented in Figure 8. The three rows correspond to the concentrated transverse load P , uniformly distributed load D , and externally applied bending moment M_o , whereas the left and right columns show Φ_W and Φ_{WG} , respectively. In most configurations, both coefficients follow the same trend with respect to the geometric parameterization. This agreement indicates that a cut that reduces the maximum displacement generally also reduces the displacement

response over the complete beam. Nevertheless, some exceptions are observed, particularly under M_o , because the two indices characterize different aspects of the displacement field.

For Case I, the results obtained under P and D exhibit the same consistent ranking for both boundary conditions and displacement coefficients, Trigonometric < Linear < Quadratic.

Thus, the trigonometric cut produces the smallest displacement penalties, the linear cut provides an intermediate response, and the quadratic cut produces the largest penalties.

For Fi-Fr under P , the trigonometric cut gives ($\Phi_W = 1.4593$) and ($\Phi_{WG} = 1.2411$), compared with (1.6297) and (1.4467) for the linear cut and (1.7736) and (1.5912) for the quadratic cut. The same ranking is observed under D , for which the trigonometric cut produces ($\Phi_W = 1.2382$) and

($\Phi_{WG} = 1.1277$). These values are lower than those obtained with the linear cut, 1.4448 and 1.3308, and the quadratic cut, 1.5896 and 1.4648.

The RS-Fr configuration follows the same tendency. Under P , the trigonometric cut produces ($\Phi_W = 1.1871$) and ($\Phi_{WG} = 1.0875$), followed by the linear cut, with 1.2527 and 1.1525, and the quadratic cut, with 1.3094 and 1.1996. Under D , the corresponding values for the trigonometric cut are $\Phi_W = 1.0864$ and $\Phi_{WG} = 1.0476$, which are again lower than those obtained with the linear and quadratic parameterizations. Therefore, for transverse concentrated and distributed loading, the boundary condition changes the magnitude of the displacement penalties but does not change the ranking among the three cut types. A different response is observed under M_o . For Fi-Fr, the linear cut minimizes the maximum-displacement coefficient, with ($\Phi_W = 2.0105$), followed by the quadratic and trigonometric cuts, with values of 2.0861 and 2.1076, respectively. However, the trigonometric cut minimizes the global-displacement coefficient, with ($\Phi_{WG} = 1.5391$), compared with 1.6664 for the linear cut and 1.7964 for the quadratic cut.

For RS-Fr under M_o , the linear cut also produces the lowest maximum-displacement coefficient, $\Phi_W = 1.4975$. The quadratic and trigonometric cuts produce nearly identical values of 1.5373 and 1.5376, respectively. Nevertheless, the trigonometric cut again provides the lowest global-displacement coefficient, $\Phi_{WG} = 1.1944$, followed by the linear and quadratic cuts, with values of 1.2561 and 1.3121. Thus, under moment loading, the parameterization that minimizes the maximum displacement does not necessarily minimize the displacement response over the entire beam.

For Case II, the quadratic cut generally produces the smallest values of both displacement coefficients. This tendency is particularly clear for Pi-Pi and RS-RS, where the same ranking is obtained for P , D , and M_o , Quadratic < Linear < Trigonometric. For Pi-Pi, the quadratic cut minimizes both coefficients under all three loads. Under P , it produces ($\Phi_W = 3.5423$) and ($\Phi_{WG} = 3.4437$), compared with 4.6409 and 4.3684 for the linear cut and 5.4859 and 5.1506 for the trigonometric cut. Under D , the quadratic values are ($\Phi_W = 3.4421$) and ($\Phi_{WG} = 3.3354$), while under M_o they are 3.0842 and 3.0190. In both cases, the linear cut produces an intermediate response and the trigonometric cut produces the largest displacement penalties.

The same tendency is observed for RS-RS. The quadratic cut gives ($\Phi_W = 2.0921$) and ($\Phi_{WG} = 1.9570$) under P , 1.9552 and 1.8511 under D , and 2.7809 and 2.7079 under M_o . These values are consistently lower than those obtained with the linear and trigonometric cuts. Therefore, for Pi-Pi and RS-RS, both displacement criteria lead to the same selection under every elemental load considered.

For Fi-Fi, the quadratic cut also provides the most favorable response in most cases. Under P , it produces ($\Phi_W = 2.2431$) and ($\Phi_{WG} = 2.0179$), while under M_o it gives 2.6906 and 2.6217. Both coefficients are lower than the corresponding values obtained with the linear and trigonometric cuts.

The only criterion-dependent exception within Case II occurs for Fi-Fi under D . In this case, the quadratic cut produces the lowest maximum-displacement coefficient, ($\Phi_W = 2.0179$), followed by the trigonometric and linear cuts, with values of 2.1059 and 2.1804. Conversely, the trigonometric cut gives the lowest global-displacement coefficient, ($\Phi_{WG} = 1.8271$), slightly below the quadratic value of 1.8418 and the linear value of 1.9110. The difference between the trigonometric and quadratic global coefficients is less than 1%, indicating that both parameterizations provide comparable global responses under this particular loading condition.

Overall, the elemental-load results show that Φ_W and Φ_{WG} generally produce the same ranking among the geometric parameterizations. For Case I under P and D , the trigonometric cut produces the lowest values of both coefficients for Fi-Fr and RS-Fr. Under M_o , the preferred parameterization depends on the displacement criterion: the linear cut minimizes (Φ_W) for both support conditions, although the quadratic and trigonometric values are nearly identical for RS-Fr, whereas the trigonometric cut minimizes (Φ_{WG}). For Case II, the quadratic cut generally produces the lowest displacement penalties and consistently minimizes both indices for Pi-Pi and RS-RS. The only relevant exception occurs for Fi-Fi under D , where the quadratic cut minimizes (Φ_W) and the trigonometric cut produces a slightly lower (Φ_{WG}).

Combined loads

The displacement-based response under combined loading conditions is presented in Figure 9. Four load combinations are considered ($D + M_o$), ($P + D$), ($P + D + M_o$) and ($P + M_o$). As in the elemental-load analysis, the left and right columns show

the maximum-displacement coefficient (Φ_w) and the global-displacement coefficient (Φ_{wG}), respectively. Since lower values of both coefficients indicate smaller displacement penalties relative to the uncut reference beam, the most favorable parameterization is the one that minimizes these indices. The results show that (Φ_w) and (Φ_{wG}) generally follow the same trend with respect to the geometric parameterization. The agreement is particularly clear for Case II, where both indices identify the quadratic cut as the most favorable alternative under every combined loading condition. For Case I, both criteria also produce the same

selection in most cases, with the only relevant differences occurring under ($D + M_o$).

For Case I, the combinations ($P + D$), ($P + D + M_o$), and ($P + M_o$) exhibit the same consistent ranking for both displacement coefficients and boundary conditions, Trigonometric < Linear < Quadratic.

Therefore, for these three load combinations, the trigonometric cut minimizes both the maximum displacement and the global displacement response, whereas the quadratic cut produces the largest penalties.

For Fi-Fr under $P + D$, the trigonometric cut produces ($\Phi_w = 1.3992$) and ($\Phi_{wG} = 1.2091$), followed by the linear cut, with values of 1.5795 and

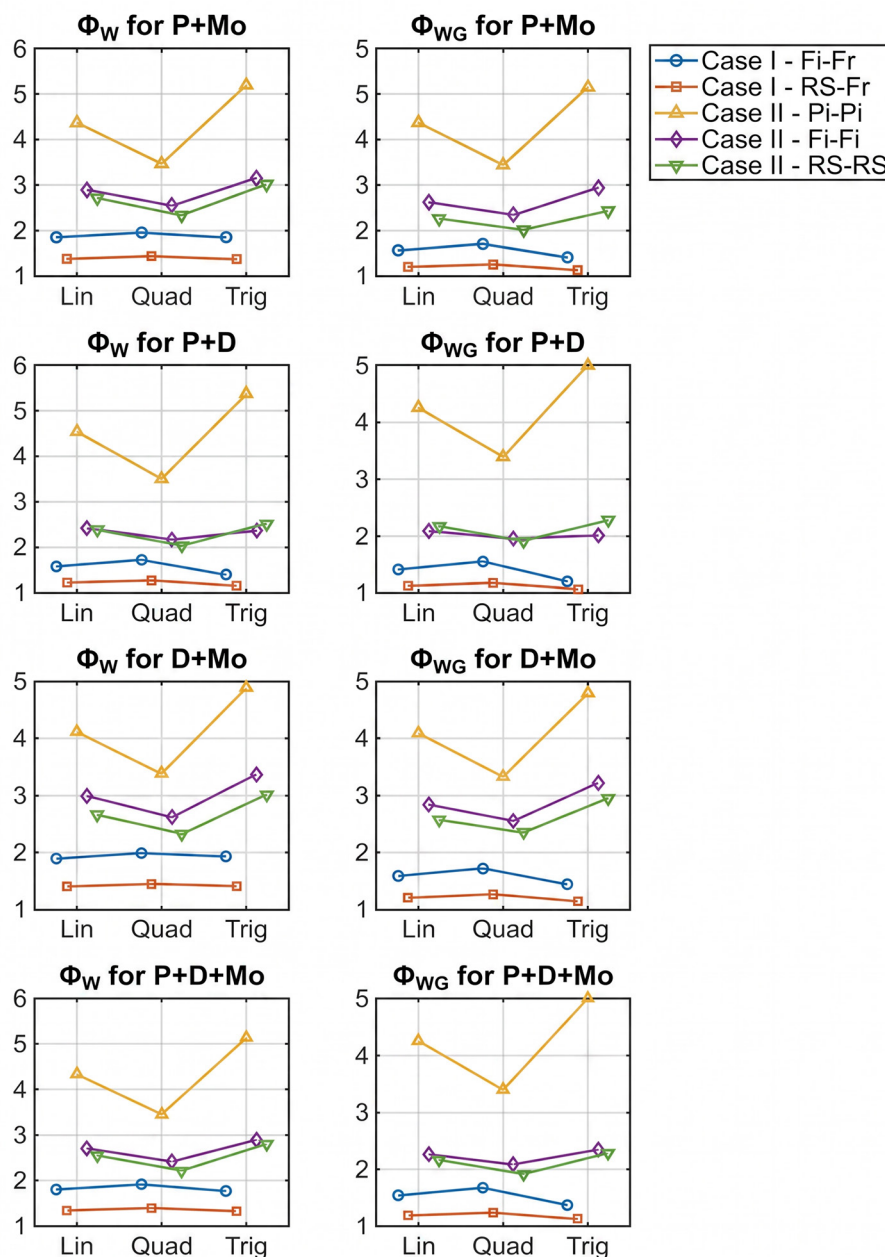


Figure 9. Displacement-based efficiency indices (Φ_w) and (Φ_{wG}) for the combined loading cases

1.4140, and the quadratic cut, with 1.7236 and 1.5555. The same ranking is observed under $P + D + M_o$, for which the trigonometric cut gives ($\Phi_w = 1.7705$) and ($\Phi_{wG} = 1.3731$), and under $P + M_o$, for which it gives 1.8495 and 1.4137. In both combinations, the linear cut produces an intermediate response and the quadratic cut produces the largest values.

The RS-Fr configuration follows the same tendency. Under $P + D$, the trigonometric cut gives $\Phi_w = 1.1559$ and $\Phi_{wG} = 1.0748$. Under $P + D + M_o$, the corresponding values are 1.3286 and 1.1264, whereas under $P + M_o$ they are 1.3782 and 1.1438. These values are consistently lower than those obtained with the linear and quadratic cuts. Thus, for these three combined loads, changing the Case I boundary condition modifies the magnitude of the displacement penalty but does not change the ranking among the parameterizations.

A criterion-dependent response is observed under $D + M_o$. For Fi-Fr, the linear cut produces the lowest maximum-displacement coefficient, $\Phi_w = 1.8987$, followed by the trigonometric and quadratic cuts, with values of 1.9356 and 1.9879, respectively. However, the trigonometric cut produces the lowest global-displacement coefficient, ($\Phi_{wG} = 1.4477$), compared with 1.5918 for the linear cut and 1.7227 for the quadratic cut.

For Case II, the quadratic cut minimizes both (Φ_w) and (Φ_{wG}) for every boundary condition and combined load considered. Consequently, both displacement criteria produce the same general design selection throughout Case II. For Fi-Fi, the combinations $D + M_o$, $P + D + M_o$, and $P + M_o$ exhibit the ranking, Quadratic < Linear < Trigonometric, for both displacement coefficients. Under $D + M_o$, the quadratic cut produces ($\Phi_w = 2.6215$) and ($\Phi_{wG} = 2.5585$), compared with 2.9925 and 2.8426 for the linear cut and 3.3702 and 3.2190 for the trigonometric cut. Under $P + D + M_o$, the quadratic values are ($\Phi_w = 2.4137$) and ($\Phi_{wG} = 2.0862$), whereas under $P + M_o$ they are 2.5468 and 2.3525.

A slightly different ranking is observed for Fi-Fi under $P + D$. The quadratic cut remains the most favorable alternative, with ($\Phi_w = 2.1680$) and ($\Phi_{wG} = 1.9570$), but it is followed by the trigonometric cut, with 2.3638 and 2.0095, and then by the linear cut, with 2.4182 and 2.0873. Although the ordering of the second- and third-best alternatives changes, both displacement coefficients continue to identify the quadratic cut as the preferred parameterization.

For Pi-Pi, the same ranking is maintained for all combined loads, Quadratic < Linear < Trigonometric. The quadratic cut produces ($\Phi_w = 3.3843$) and ($\Phi_{wG} = 3.3354$) under $D + M_o$, 3.5037 and 3.4014 under $P + D$, 3.4516 and 3.4014 under $P + D + M_o$, and 3.4719 and 3.4437 under $P + M_o$. In every case, the linear cut produces intermediate values and the trigonometric cut produces the largest displacement penalties.

The RS-RS configuration follows the same tendency. The quadratic cut gives ($\Phi_w = 2.3299$) and ($\Phi_{wG} = 2.3501$) under $D + M_o$, 2.0407 and 1.9164 under $P + D$, 2.2196 and 1.9164 under $P + D + M_o$, and 2.3468 and 2.0205 under $P + M_o$. These values are consistently lower than those obtained with the linear and trigonometric parameterizations.

Overall, the combined-load results show a clear distinction between Case I and Case II. For Case I, the trigonometric cut minimizes both (Φ_w) and (Φ_{wG}) under $P + D$, $P + D + M_o$, and $P + M_o$ for Fi-Fr and RS-Fr. Under $D + M_o$, the linear cut minimizes (Φ_w), whereas the trigonometric cut minimizes (Φ_{wG}). Therefore, the trigonometric parameterization remains the most consistent alternative when both maximum and global displacement criteria are considered.

For Case II, the quadratic cut minimizes both displacement coefficients for every combined load and boundary condition investigated. This tendency is especially clear for Pi-Pi and RS-RS, where the ranking quadratic-linear-trigonometric remains unchanged. For Fi-Fi, the quadratic cut also remains the preferred alternative, although the relative ordering of the linear and trigonometric cuts changes under $P + D$. These results confirm that (Φ_w) and (Φ_{wG}) generally provide consistent evaluations under combined loading, with criterion-dependent selections occurring only for Case I under $D + M_o$.

Mechanical interpretation of the structural response

The three coefficients provide complementary information: (Φ_w) measures the maximum local displacement, (Φ_{wG}) represents the global displacement response, and (Φ_o) reflects the balance between the loss of flexural stiffness and the reduction in structural mass. Since all cut parameterizations retain the same volume, their different responses arise primarily from how the retained material is distributed relative to the

bending-moment distribution and boundary conditions. Removing material from regions of high bending moment produces larger curvature and displacement penalties, whereas removal from less mechanically influential regions can reduce mass with a smaller loss of stiffness.

For Case I, the trigonometric cut provides the most balanced response. Its smooth variation preserves greater rigidity near the constrained end and concentrates more of the section reduction toward the free end. Consequently, it produces the highest frequency ratios, minimizes (Φ_{WG}) for every load, and minimizes (Φ_w) in most cases. Under M_o and $D + M_o$, the linear cut produces the lowest (Φ_w) , whereas the trigonometric cut continues to minimize (Φ_{WG}) . This difference indicates that the parameterization minimizing the displacement peak does not necessarily minimize the deformation over the complete beam.

For Case II, the quadratic cut generally produces the lowest displacement penalties because its greater central section height preserves more flexural rigidity within the span. It also provides the highest frequency ratios for Pi-Pi and RS-RS. Fi-Fi presents a trade-off: the quadratic cut is generally preferable for displacement control, whereas the trigonometric cut better preserves the fundamental frequency. Overall, the similar rankings obtained from (Φ_w) and (Φ_{WG}) show that the cut geometry usually affects the local and global displacement responses consistently, with only a limited number of criterion-dependent exceptions.

Practical selection criteria

For Case I, the trigonometric cut is recommended as the most robust alternative because it provides the highest frequency ratios and the lowest global displacement penalties for every elemental and combined load investigated. It also minimizes the maximum-displacement coefficient in most cases. The linear cut may be considered when the design is governed exclusively by (Φ_w) under M_o or $D + M_o$. For Case II, the quadratic cut is generally recommended for displacement control. It minimizes both (Φ_w) and (Φ_{WG}) for every elemental and combined load applied to Pi-Pi and RS-RS and for nearly all Fi-Fi conditions. The only relevant exception occurs for Fi-Fi under D , where the quadratic cut minimizes (Φ_w) and the trigonometric cut produces a slightly lower (Φ_{WG}) . According to the frequency-based criterion, the trigonometric cut is preferable for Fi-Fr, RS-Fr,

and Fi-Fi, whereas the quadratic cut is preferable for Pi-Pi and RS-RS. Therefore, the static and frequency-based criteria lead to the same general recommendation for all configurations except Fi-Fi, for which the final selection depends on whether displacement control or fundamental-frequency preservation is the dominant design objective.

CONCLUSIONS

This work presented a comparative study of lightweight non-prismatic beams with linear, quadratic, and trigonometric cross-sectional variation laws under elemental and combined loading conditions. All configurations were analyzed under an equal-volume criterion, allowing the influence of the stiffness distribution to be evaluated independently from the total amount of material retained. The structural response was assessed using a Euler-Bernoulli finite element formulation and three dimensionless indices associated with maximum displacement (Φ_w) , global displacement (Φ_{WG}) , and fundamental natural frequency (Φ_ω) .

The results demonstrate that structural performance depends on the interaction between the spatial distribution of flexural rigidity, the bending-moment distribution, and the boundary condition. Although all reduced configurations retain the same structural volume, they distribute material differently along the beam. Parameterizations that preserve greater rigidity in regions subjected to high bending moments produce smaller curvature, strain-energy accumulation, and displacement penalties. The location of the removed material also determines whether the reduction in mass compensates for the loss of stiffness in the fundamental-frequency response.

According to the frequency-based criterion, the trigonometric cut provides the highest values of (Φ_ω) for Fi-Fr, RS-Fr, and Fi-Fi, whereas the quadratic cut is preferable for Pi-Pi and RS-RS. The Case I frequency ratios show that an appropriate material distribution can increase the fundamental frequency relative to the uncut beam. In contrast, all Case II frequency ratios remain below unity, indicating that the stiffness reduction dominates the mass-reduction effect in these configurations.

For the displacement-based response, all reduced-section configurations produce $\Phi_w > 1$ and $\Phi_{WG} > 1$, indicating larger displacements than the uncut reference beam. Nevertheless, a clear distinction is observed between the geometric families.

For Case I, the trigonometric cut generally produces the smallest displacement penalties and minimizes (Φ_{WG}) under every elemental and combined load considered. The linear cut minimizes (Φ_W) in the moment-related cases M_o and $D + M_o$. For Case II, the quadratic cut generally minimizes both displacement coefficients. The only relevant exception occurs for Fi-Fi under D , where the trigonometric cut produces a slightly lower (Φ_{WG}).

From a practical perspective, the trigonometric cut is recommended for cantilever-like Case I configurations, particularly when global displacement and fundamental-frequency preservation are relevant. The quadratic cut is recommended for Case II configurations governed by displacement control and is also the preferred frequency-based alternative for Pi-Pi and RS-RS. For Fi-Fi, the final selection involves a trade-off between the quadratic cut for displacement control and the trigonometric cut for frequency preservation. These results provide mechanics-based selection criteria for preliminary beam design and can complement more advanced geometry-optimization procedures.

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