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Type A and B errors as a basis for determining the uncertainty of algorithms for processing of measurement data series

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ABSTRACT

Sequences of samples of time-varying quantities processed by an algorithm are burdened by errors that vary randomly in successive samples (type A) and errors that have constant values (type B). These errors propagate differently through the algorithm, and therefore their specifics must be taken into account in the process of determining the uncertainty of the result at the output of the algorithm. The article describes a mathematical apparatus that allows the description of the result in the form of a measurand interval formed on the basis of a probabilistic description of both types of error. Analytical considerations are illustrated with examples.

Keywords: processing algorithm, measurement error, measurand, measurement uncertainty.

INTRODUCTION

Processing of sequences of measurement data is widely used in modern instruments and measurement sstems, especially when the measured quantity changes over time. In this case, the data processed by an algorithm is a sequence of samples $x_0, x_1, \ldots, x_{K-1}$ of the quantity $x(t)$ covered by the measurement window as shown in Figure 1.

The numerical operations performed on a sequence of samples are determined by a linear or non-linear processing algorithm [1], which can process data sequences from more than one measurement window. Regardless of the type of algorithm, the analysis of its metrological properties and the determination of the uncertainty of the result obtained at its output requires the use of a simple and coherent mathematical apparatus linking the errors burdening the samples and the errors introduced by the algorithm itself. The description of errors in probabilistic categories enables the determination of uncertainty at the output of algorithms in complex measurement situations, when there are both static and dynamic errors arising in non-linear sensors [2].

Currently, a probabilistic approach to determining measurement result uncertainty is widely used, based on the assumptions presented in document [3], which, after numerous supplements, took the forms [4, 5] applied the terms included in the vocabulary [6]. Such a type of this 'classical' approach has been used to determine the uncertainty of results obtained in mechanical measurements [7], as well as the results of algorithmic processing of measurement data sequences [8]. The basis of the procedure for describing measurement result inaccuracy proposed in documents [4, 5] is the assumption of uncertainty as a primary quantity, but without its formal definition. This approach offers a wide range of possibilities for combining various factors influencing measurement result inaccuracy in laboratory conditions. However, it is very difficult to apply in the case of an algorithm for processing measurement data sequences, because it causes the resulting procedures to be mathematically complex [8]. Another reason for the difficulty in applying this type of procedure to determine the uncertainty of an algorithm's output result is its intended use in stable laboratory conditions [7], where multiple measurements

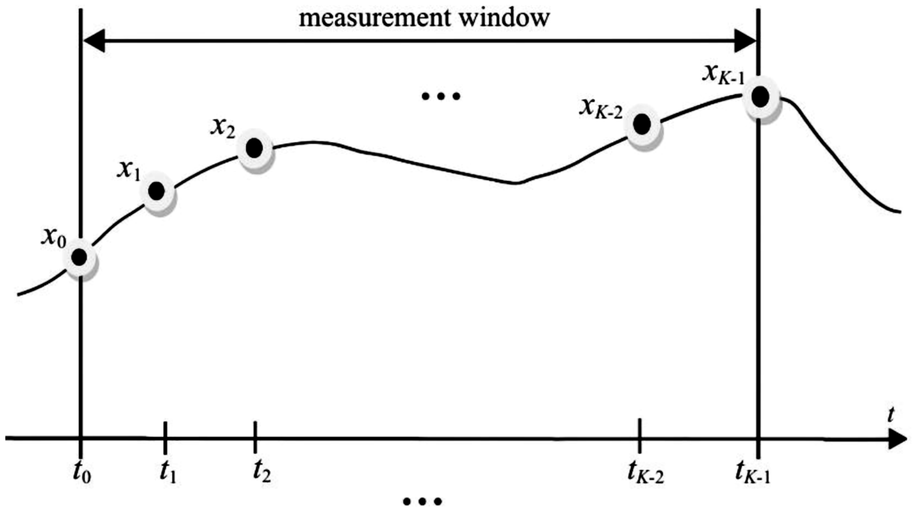

Figure 1. Measurement window of the time-varying input quantity of the processing algorithmt_0, t_1, …, t_{K-1} – sampling moments

of the same quantity are possible. The algorithm described, however, processes the measurement results of instantaneous values of a time-varying input quantity, which are inherently unique.

The literature on the subject offers alternative methods for determining measurement result uncertainty, using the Bayesian approach [9] and fuzzy sets [10]. However, these methods cannot be used for algorithms processing measurement data sequences, as they are characterized by a complex mathematical formalism. The algorithm performs numerous arithmetical operations on measurement results that are burdened by many errors, making determining the uncertainty of the algorithm's output inherently complex and requiring the use of simple probabilistic methods for describing error sources, such as those proposed in this work.

The mathematical apparatus presented in the article uses the mathematical definition of uncertainty introduced in [11] and measurement error obtained as a result of the analysis of the measurement process using a standard with a quantum (granular) structure [2]. After transforming the error definition, a model of measurand is obtained, which is the basis for the analysis of the metrological properties of processing algorithms. The key to this analysis is the term "measurand" used here in a more strict sense than in the vocabulary [6]. This is related to the fact that for time-varying quantities, each sample in the measurement window has a different value, and thus must be individually treated as a measurand defined in such a case as an instantaneous symptom of the measured quantity. Thus, an important distinction emerges between the measured quantity, for example, voltage, and the measurand, which is a sample of that voltage. In this view, the algorithm converts a sequence of input measurands, defined by the measurement window, into a single output measurand.

The numerical complexity of the algorithm means that the most effective way to determine the error distribution of its output result is to use the Monte Carlo method [12], which is recommended in the procedures applied for determining uncertainty [13]. The simulation-based nature of this method means that its results must be verified. The mathematical apparatus presented in this paper provides the means to enable such verification.

METROLOGICAL DECOMPOSITION OF THE ALGORITHM

The starting point of the algorithm's decomposition for metrological analysis is the definition of the error e of the measurement result using a standard with a quantum structure [2]

$$e = x - \hat{x} \tag{1}$$

where: x is the unknown real (true) value of the measurand, and $\hat{x}$ is the estimate of the result of its measurement obtained by removing systematic errors from the raw result. This means that the estimate is burdened only by unremovable random errors, the sum of which is the total error

$$e = e_1 + e_2 + \ldots + e_J \quad (2)$$

where: J is the number of random partial errors. Thus, the estimate can be defined as the number closest to the real value of the measurand from the numbers possible under the specified measurement conditions.

After transforming expression (1), the measurand model is obtained as

$$x = \hat{x} + e \quad (3)$$

whose interpretation in probabilistic terms means that the population of the measurand estimate differs from the population of the estimate error by a constant real value of the measurand. This leads to the conclusion that the uncertainty of the measurand after its measurement, treated as a parameter characterizing the dispersion of the measurement results [6], can be determined on the basis of the total error distribution of the estimate obtained at the output of the algorithm.

The subject of further consideration is the linear algorithm

$$z = a_0 x_0 + a_1 x_1 + \ldots + a_{K-1} x_{K-1} \quad (4)$$

which is a linear combination of input measurands x_0, x_1, …, x_{K-1} and coefficients with constant values a_0, a_1, …, a_{K-1}, z is the output measurand.

After introducing the measurand model (3) into the algorithm (4), the following expression is obtained

$$\hat{z} + e = a_0(\hat{x}_0 + e_0) + a_1(\hat{x}_1 + e_1) + \ldots + + a_{K-1}(\hat{x}_{K-1} + e_{K-1}) \quad (5)$$

which can be decomposed into two relationships. The first is the equation that describes processing the estimate

$$\hat{z} = a_0 \hat{x}_0 + a_1 \hat{x}_1 + \ldots + a_{K-1} \hat{x}_{K-1} \quad (6)$$

according to which the value of the output estimate is determined as the linear combination of K input estimates.

The second relationship

$$e = a_0 e_0 + a_1 e_1 + \ldots + a_{K-1} e_{K-1} \quad (7)$$

is called the error propagation equation and describes the relationships linking the error of the output estimate to the errors of the input estimates. It can be used to analyze the relationship between errors, to determine equations describing the propagation of errors from the input to the output of the algorithm, as well as to calculate the uncertainty of the output estimate when the total error of this estimate has a distribution close to normal.

PROPAGATION OF TYPE A AND B ERRORS BY AN ALGORITHM

The propagation of errors by an algorithm consists in changing their values due to the execution of numerical operations specific to the algorithm. The description of propagation consists in giving the relations that connect the realizations of a certain type of error at the output and input of the algorithm, that is, in determining the relations between the errors of the input measurand estimates and the errors of the output estimate. The basic way to obtain this description is to use the error Equation 7. The analysis of this equation shows that, from the point of view of propagation, two types of errors should be distinguished: those with values that vary randomly in successive samples (measurands) covered by the measurement window, and those with the constant values in the samples. By analogy with the two types of uncertainty defined in the vocabulary [6], these errors are called A-type and B-type errors, respectively [11].

Propagation of type A errors by the algorithm

The propagation of type A errors varying randomly in the samples from a measurement window can be determined by the relationships between the standard deviations of the same error at the input and output of the algorithm. Assuming that the measurements for all windows are performed under the same conditions, each input estimate is subject to the realization of the same error with a standard deviation generally denoted as $\sigma_{\text{ran,in}}$. In this case, according to relation (7), the standard deviation $\sigma_{\text{ran,out}}$ of this error in the output of the algorithm is described by the expression

$$\sigma^2_{\text{ran,out}} = (a_0 \sigma_{\text{ran,in}})^2 + (a_1 \sigma_{\text{ran,in}})^2 + \ldots + + (a_{K-1} \sigma_{\text{ran,in}})^2 = (a_0^2 + a_1^2 + \ldots + a_{K-1}^2)\sigma^2_{\text{ran,in}} \quad (8)$$

According to it, the properties of the algorithm during the processing of errors that vary randomly in the measurement window are described by the random error propagation coefficient, which has the form:

$$k_{ran} = \frac{\sigma_{ran,out}}{\sigma_{ran,in}} = \sqrt{a_0^2 + a_1^2 + \ldots + a_{K-1}^2}. \quad (9)$$

Algorithms for which the value of the propagation coefficient is less than 1 perform filtering of random errors (reduce their value), and when it is greater than 1 random errors are amplified, as happens in the case of dynamic reconstruction algorithms [2]. In the following considerations, the exemplary algorithm for determining the average value of a sinusoidal waveform is used. The basis of the algorithm is the definition

$$x_{av} = \frac{1}{T}\int_0^T x(t)\mathrm{d}t \quad (10)$$

where: T is the period of the waveform of a time-varying quantity $x(t)$.

This waveform is sampled with a period T_s in a measurement window with a width equal to the period T of the sine wave. According to the applied algorithm calculating average value (10), the waveform of the quantity between sampling moments is approximated by a segment of a horizontal straight line starting from the sample value. This algorithm is described by the expression

$$x_{av} = \frac{1}{T}\left(T_s x_0 + T_s x_1 + \ldots + T_s x_{K-1}\right) =$$

$$= \frac{T_s}{T}\sum_{k=0}^{K-1} x_k = a\sum_{k=0}^{K-1} x_k \quad (11)$$

where: x_0, x_1,…, x_{K-1} are the algorithm's input measurands (samples), and the average value of the waveform x_{av} is the output measurand.

After decomposing the relation (11) as described in Section 2, the Equation 6 described processing the estimates is obtained in the form

$$\hat{x}_{av} = a\sum_{k=0}^{K-1} \hat{x}_k \quad (12)$$

and the error propagation Equation 7

$$e = a\left(e_0 + e_1 + \ldots + e_{K-1}\right) \quad (13)$$

All coefficients of the algorithm have the same values, so that the propagation coefficient (9) of the A-type error in this case has the form

$$k_A = k_{ran} = \sqrt{a_0^2 + a_1^2 + \ldots + a_{K-1}^2}. = \sqrt{Ka^2} = a\sqrt{K} \quad (14)$$

Example 1. The number of samples in the measurement window with a width T equal to the period of the voltage waveform is $K = 100$, which means that the sampling period is equal to $T_s = 0.01\ T$, so the coefficient in Equation 14 has the value $a = T_s/T = 0.01$. The samples are quantized by using a 12-bit unipolar A/D converter with a quantum value $q = 1$ mV. The quantization error is described by a rectangular distribution in the range from $-q/2$ to $q/2$ [2], so the standard deviation of this error at the input of the algorithm is

$$\sigma_{q,in} = \frac{q}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.29\,\mathrm{mV} \quad (15)$$

According to formula (14), the propagation coefficient of the random error for the given parameters of the algorithm has the value of

$$k_A = a\sqrt{K} = 0.01\sqrt{100} = 0.1 \quad (16)$$

which means that random errors are suppressed (filtered) by the algorithm by a factor of 10, and therefore, based on relations (9) and (16), the standard deviation of the quantization error in the output of the algorithm is

$$\sigma_{q,out} = k_A \sigma_{q,in} = 0.1 \cdot 0.29\,\mathrm{mV} = 0.029 \cdot 10^{-3}\ \mathrm{V} \quad (17)$$

Propagation of type B errors by the algorithm

Based on relation (7), the propagation of B-type errors with constant values in the measurement window, generally denoted as $e_{c,in}$, is described by the expression

$$e_{c,out} = a_0 e_{c,in} + a_1 e_{c,in} + \ldots + a_{K-1} e_{c,in} =$$

$$= e_{c,in}\left(a_0 + a_1 + \ldots + a_{K-1}\right) \quad (18)$$

according to which the propagation coefficient for this type of error is

$$k_B = \frac{e_{c,out}}{e_{c,in}} = a_0 + a_1 + \ldots + a_{K-1} \quad (19)$$

For the exemplary algorithm, this coefficient is of the form

$$k_B = a_0 + a_1 + \ldots + a_{K-1} = Ka \quad (20)$$

Type B errors can be divided into additive errors, that is, errors that propagate independently of the values of the estimates processed by the algorithm, and multiplicative errors, which depend on these estimates. The propagation of additive error is described in Example 2.

Example 2. As calculated in Example 1, the coefficients of the example algorithm (11) have the same values $a = 0.01$ for $K = 100$ samples in the measurement window. According to formula (20), the coefficient of propagation of additive B-type errors for this algorithm takes the value of

$$k_B = Ka = 100 \cdot 0.01 = 1 \tag{21}$$

which means that errors of this type are transferred from the input to the output of the algorithm without changing the value.

Most often, B-type errors are caused by temperature changes. For example, this type of error is related to the zero drift of the A/D converter used to measure samples, assuming that the temperature is constant during the measurement window. Let us assume that the input estimate burdened by the additive drift error is described as $\hat{x} = x + \varepsilon_{sh} \Delta\vartheta$, where ε_{sh} is the coefficient of sensitivity of the converter's zero to changes in temperature. Accordingly with Equation 1 and this assumption, the drift error is described by the expression

$$e_{sh,in} = x - \hat{x} = x - (x + \varepsilon_{sh}\Delta\vartheta) = -\varepsilon_{sh}\Delta\vartheta \tag{22}$$

where: $\Delta\vartheta = \vartheta - \vartheta_0$ is the change of the temperature ϑ relative to the reference temperature ϑ_0.

Assuming that ϑ_0 = 25 °C, and the temperature can vary from 5 °C to 45 °C, the maximum absolute temperature change is $\Delta\vartheta_{max}$ = 20 °C, which means that for the sensitivity factor $\varepsilon_{sh} = 10^{-5}$ V/°C the maximum zero error at the output of the algorithm is

$$|e_{sh,out}| = k_B e_{sh,in} = k_B \varepsilon_{sh} \Delta\vartheta = 1 \cdot 10^{-5} \cdot 20 = 0.2 \cdot 10^{-3}\ \text{V} \tag{23}$$

When the ambient temperature is measured, the value of the zero offset error can be calculated according to formula (22) and can be removed from the processing result as a correction. On the other hand, when the temperature is not controlled, it can be assumed that its value changes randomly within the range and then the error $e_{sh,out}$ at the output of the algorithm should be considered as random with a rectangular distribution in the range of $-0.2 \cdot 10^{-3}$ do $0.2 \cdot 10^{-3}$ V.

An exemplary multiplicative type B error is caused by temperature changes of the quanta of the A/D converter. Assuming that the values of all quanta change linearly as a function of temperature increment $\Delta\vartheta$ relative to the nominal value ϑ_0, the value of a single quantum, constant for the measurement window is described by the expression

$$\tilde{q} = (1 + \varepsilon_{inc}\Delta\vartheta)q, \quad \Delta\vartheta = \vartheta - \vartheta_0 \tag{24}$$

where: q is the nominal value of the quantum, and ε_{inc} is the coefficient of sensitivity of the quantum to the temperature changes.

Given that changes in quantum values cause a change in the slope of the A/D converter's characteristic, this coefficient determines the temperature dependence of the slope of this characteristic.

The occurrence of multiplicative errors is a feature of any measuring instrument, so each type of this error should be considered in a specific way for a particular instrument determining its metrological properties in a way that is not related to the type of algorithm processing the data obtained with this instrument.

For an A/D converter whose quantum varies in the manner described by relation (24), the result of a single quantization can be described by the relation [2]

$$\hat{x} = q\,\text{ent}\left(\frac{x}{\tilde{q}}\right) = q\,\text{ent}\left(\frac{x}{(1+\varepsilon_{inc}\Delta\vartheta)q}\right) \cong$$
$$\cong q\frac{x}{(1+\varepsilon_{inc}\Delta\vartheta)q} = \frac{x}{1+\varepsilon_{inc}\Delta\vartheta} \tag{25}$$

in which the entier function (denoted by 'ent') determines the integer part of its argument. In order to eliminate the quantization error in expression (25), this function is replaced by its linear approximation. Taking this into account and according to definition (1) and expression (25), the error of a single measurement with an A/D converter caused by a change in the slope of the characteristic is of the form

$$e_{inc} = x - \hat{x} = x - \frac{x}{1+\varepsilon_{inc}\Delta\vartheta} \cong$$
$$\cong x - x(1 - \varepsilon_{inc}\Delta\vartheta) = x\varepsilon_{inc}\Delta\vartheta \tag{26}$$

obtained with assumption that ε_{inc} makes the denominator close to 1.

In order to determine a probabilistic description of this error for an exemplary A/D converter, Experiment 1 was performed using the Monte Carlo method.

Experiment 1. The distribution of the measurement error due to temperature changes of the slope of the characteristic caused by changes in the value of the quantum of the exemplary 12-bit A/D converter with a quantum value of q = 1 mV, whose measurement range is from 0 to 2^{12} $1\cdot10^{-3}$ V = 4.096 V, is determined**.** Quantization error is neglected. The experiment is carried out in the following steps.

Determine the environmental temperature ϑ of the converter as random in the range from 5 to 45 °C according to a rectangular distribution.

Determine value of the measured voltage x as random in the range from 0 to 4.096 V according to the rectangular distribution.

Calculate the error value according to the relation (26)

$$e_{\text{inc}} = -x\varepsilon_{\text{inc}}\Delta\vartheta, \quad \varepsilon_{\text{inc}} = 10^{-5}\,\frac{1}{^{\circ}\text{C}} \tag{27}$$

and place the result in the set of the error values.

Repeat steps (1) through (3) 100,000 times. Present the obtained set of error values in the form of a histogram.

The error distribution shown in Figure 2 describes a property of the exemplary A/D converter as a measurement instrument. Errors of this kind burden the measurands processed by the algorithm, and therefore the error in its output caused by the propagation of these errors must be considered for a specific algorithm. According to relation (26), the input estimate of the number k from the measurement window is burdened by the error

$$e_{\text{inc},k} = \hat{x}_k\varepsilon_{\text{inc}}\Delta\vartheta \tag{28}$$

According to relation (13), the values of this error in the measurement window add up, so the multiplicative B-type error in the output of the exemplary algorithm is described by the expression

$$e_{\text{inc,out}} = \sum_{k=0}^{K-1} a e_{\text{inc},k} = \varepsilon_{\text{inc}}\Delta\vartheta\sum_{k=0}^{K-1} a\hat{x}_k = \varepsilon_{\text{inc}}\Delta\vartheta\,\hat{x}_{\text{av}} \tag{29}$$

In Equation 29 there is used relation (12), according to which the estimate of the output measurand is a weighted sum of the input measurands. Expression (29) means that the described type B multiplication error is proportional to the value of the output measurand.

Example 3. The temperature coefficient of changes of the characteristic slope has a value of ε_{inc} = 10^{-5} 1/V°C and the estimate of the mean value is $\hat{x}_{\text{av}}$ = 2 V. For the temperature changes in the range of $\Delta\vartheta$ = 20 °C in relation to its nominal value and, the maximum value of the error at the output of the algorithm caused by the temperature according to formula (29) is

$$\left|e_{\text{inc,out}}\right| = 10^{-5}\cdot 20\cdot 2 = 0.4\cdot10^{-3}\ \text{V} \tag{30}$$

Taking value (30) into account and for random temperature changes in successive measurement

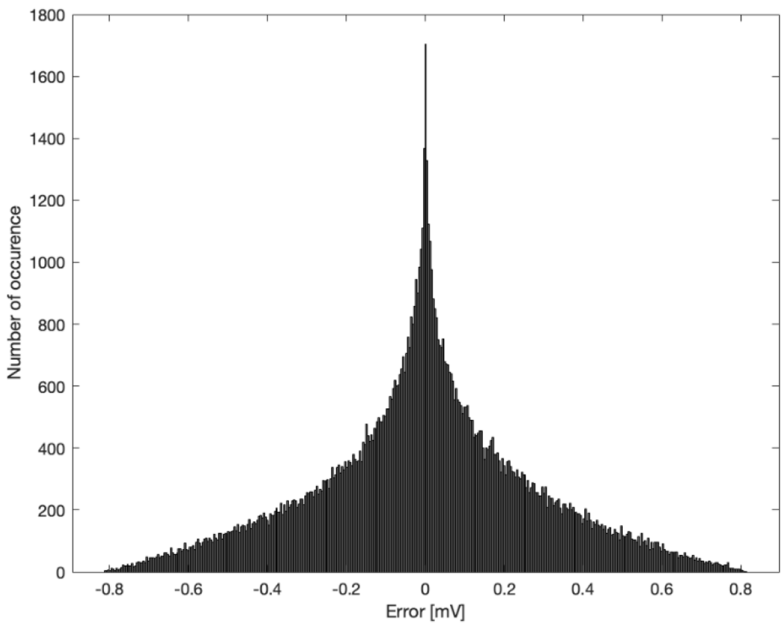

Figure 2. Histogram of the error caused by temperature changes of the characteristic slope of the exemplary A/D converter due to a change of the quantum value

windows, the slope error can be described by a rectangular distribution in the range of -0.4 to 0.4 mV. However, this error is highly correlated with the zero drift error due to the temperature dependence of both errors. For this reason, it is simpler to use the sum of both errors at the output of the algorithm based on equation (7). According to Equations 21, 22 and 29 we get

$$e_{\text{temp,out}} = e_{\text{sh,out}} + e_{\text{inc,out}} = -k_{\text{B}}\varepsilon_{\text{sh}}\Delta\vartheta + \varepsilon_{\text{inc}}\Delta\vartheta\hat{x}_{\text{a}}, \\ = \left(\varepsilon_{\text{inc}}\hat{x}_{\text{av}} - \varepsilon_{\text{sh}}\right)\Delta\vartheta \quad (31)$$

Equation 31 describes the total temperature error at the output of the algorithm. According to this relationship, the maximum value of this error for the taken parameters and operating conditions of the A/D converter is

$$\left|e_{\text{temp,out}}\right| = \left|\left(2\cdot10^{-5} - 1\cdot10^{-5}\right)\cdot 20\right| = 0.2\,\text{mV} \quad (32)$$

For random temperature variations within the given limits, the error (32) can be described by a uniform distribution with a range from -0.2 to 0.2 mV. In this case, the standard deviation of this error has a value of

$$\sigma_{\text{temp,out}} = \frac{0.2}{\sqrt{3}} = 0.115\,\text{mV}. \quad (33)$$

DETERMINATION OF THE UNCERTAINTY INTERVAL OF THE MEASURAND AT THE OUTPUT OF THE ALGORITHM

As a result of performing the algorithm for processing measurement data, a measurand is obtained, the inaccuracy of which must be determined in a generally acceptable way. An interval description of the measurement result is commonly used for this purpose [4]. In the paper, the inaccuracy of the algorithm's output measurand is determined by means of a measurand interval, defined as such a numerical interval in which the real measurand value is located with a taken probability [8]. The starting point for determining this interval is the formal definition of the uncertainty interval in the form of the expression

$$\text{P}\left[\underline{u} \le \left(x - \hat{x}\right) \le \overline{u}\right] = p \quad (34)$$

according to which the probability **P** of appearance of the difference between the actual value of the measurand x and its estimate within an uncertainty interval with lower $\underline{u}$ and upper $\overline{u}$ limits is p.

The difference $x - \hat{x}$ is the definition (1) of the estimation error, which means that according to expression (34), the probability of occurrence of the error realization in the uncertainty interval is p. This statement allows us to determine the limits of the uncertainty interval on the basis of the probability density function $g(e)$ of the error e using the following functionals:

$$\int_{-\infty}^{\underline{u}} g(e)\,\text{d}e = \frac{1-p}{2}, \qquad \int_{\overline{u}}^{\infty} g(e)\,\text{d}e = \frac{1-p}{2} \quad (35)$$

The limits of the uncertainty interval can be calculated analytically, as in the following example, or on the basis of a histogram, as shown using the results of Experiment 2.

Example 4. The probability density function of the quantization error has the form [2]

$$g(e) = \begin{cases} \frac{1}{q} & \text{for } -\frac{q}{2} \le e \le \frac{q}{2} \\ 0 & \text{for the others} \end{cases} \quad (36)$$

Based on expressions (35), the limits of the uncertainty interval are determined. For confidence level p = 0.95 and quantum value q = 1 mV, the lower limit obtained after solving the functional

$$\int_{-\frac{q}{2}}^{\underline{u}} \frac{1}{q}\,\text{d}e = \frac{1-p}{2}, \quad (37)$$

takes the value

$$\underline{u} = \frac{-pq}{2} = \frac{-0.95\cdot 1}{2} = -0.475\text{ mV} \quad (38)$$

The upper limit of the interval calculated in the same way is

$$\overline{u} = \frac{pq}{2} = \frac{0.95\cdot 1}{2} 0.475\text{ mv} \quad (39)$$

Therefore, the uncertainty interval has the form

$$\vec{u} = \left[\underline{u};\overline{u}\right] = \left[-0.475;\ 0.475\right]\text{mV} \quad (40)$$

Experiment 2. The purpose of the experiment is to determine the uncertainty interval of the total error of the measurand at the output of the exemplary algorithm for input measurands obtained by quantizing noised samples and burdened by the described type B errors. The experiment consists of the following steps.

1. The 12-bit A/D converter operates at a temperature ϑ constant during the measurement window, and for subsequent windows varying randomly according to the rectangular distribution in the range from od 5 do 45 °C. Determine the temperature value and calculate the quantum value for a single measurement window according to relation (24) as

$$\tilde{q} = (1 + \varepsilon_{\text{inc}} \Delta\vartheta) q, \quad \Delta\vartheta = \vartheta - \vartheta_0,$$

$$q = 1\,\text{mV}, \ \vartheta_{\text{o}} = 25\ ^{\circ}\text{C}, \ \varepsilon_{\text{inc}} = 1 \cdot 10^{-5} \frac{1}{^{\circ}\text{C}} \quad (41)$$

and, based on expression (22), the zero drift from equation

$$v_{\text{sh}} = \varepsilon_{\text{sh}} \cdot \Delta\vartheta \quad (42)$$

where: the drift factor is $\varepsilon_{sh} = 10^{-5}$ V/°C.

2. In the measurement window of width $T = 1$ s determine 100 equidistant voltage samples x_k, $k = 0, 1, \ldots, 99$ with period T varying sinusoidally in the range from 1 to 3 V, whose phase with respect to the beginning of the measurement window is determined randomly in the range from 0 to T according to a rectangular distribution.
3. For the samples determined in step 2 and the values of quantum and zero drift error determined in step 1 and 2, respectively, calculate 100 input voltage estimates according to the relation

$$\hat{x}_k = q\text{ent}\left[\frac{x_k + v_{\text{noi}} + v_{\text{sh}}}{\tilde{q}} + 0.5\right], \quad k = 0, 1, \ldots, 99 \quad (43)$$

where: v_{noi} is the voltage noise with a normal distribution of N(0, 1) mV, q is the nominal quantum value equal to 1 mV.

4. Determine the value of the output estimate of the exemplary algorithm by summing 100 calculated input estimates and multiplying the sum by a factor $a = 0.01$.
5. Calculate the error of the output estimate according to the error definition (1) with respect to the real average value calculated from definition (10) and equal $x_{av} = 2$ V, then place the result in the set of error values.
6. Repeat steps 1 through 5 100,000 times.
7. Present the resulting set of error values in the form of a histogram and determine the standard deviation and the limits of the uncertainty interval for $p = 0.95$.

Having known the uncertainty interval, one can determine the measurand interval. According to the definition (34), the following inequality is satisfied with probability p

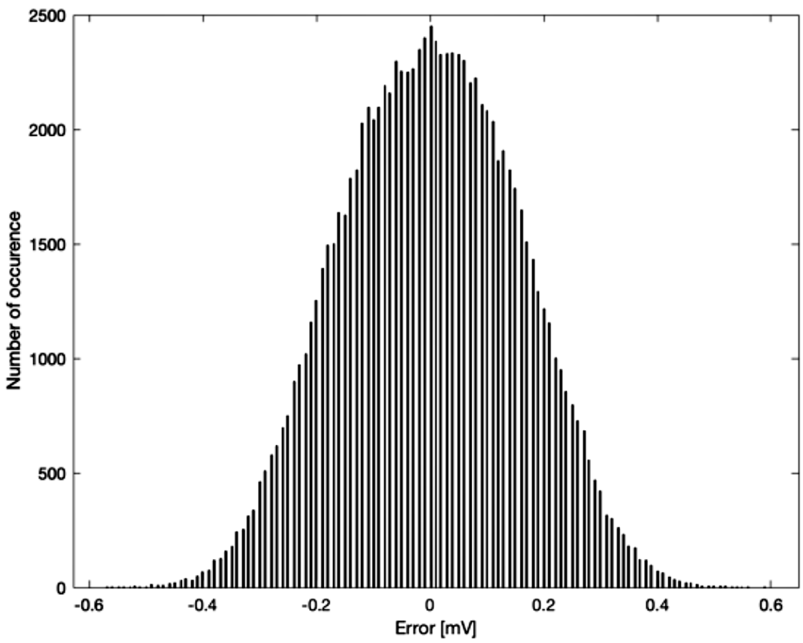

Figure 3. Histogram of the total error in the output of the exemplary algorithm, the standard deviation of the error is 0.156 mV, the limits of the uncertainty interval calculated according to expressions (35) for the confidence level $p = 0.95$ have the values $\underline{u} = -0.30\,\text{mV}$ and $\overline{u} = 0.30\,\text{mV}$

$$\underline{u} \le (x - \hat{x}) \le \overline{u} \quad (44)$$

after transformation of which the expression is obtained

$$\underline{u} + \hat{x} \le x \le \overline{u} + \hat{x} \quad (45)$$

Inequality (45) means that with probability p the real value of the measurand is contained in the measurand interval described as:

$$\vec{x} = [\hat{x} + \underline{u} \; ; \; \hat{x} + \overline{u}] \quad (46)$$

According to the rules of interval arithmetic [14], this interval can be written in the form

$$\vec{x} = \hat{x} + [\underline{u} \; , \; \overline{u}] \quad (47)$$

i.e., as the sum of the measurand estimate and the uncertainty interval with limits determined from the distribution of the total error of the estimate according to the functionals (35).

Example 5. For the first measurement window determined in Experiment 2, 100 values of the input measurand estimates were obtained, based on which the output estimate of the example algorithm was calculated equal to $\hat{x}_{av} = 1.99979$ V. As stated in the Figure 3 caption, the limits of the uncertainty interval are $\underline{u} = -0.30\,\text{mV}$ and $\overline{u} = 0.30\,\text{mV}$. According to the expressions (46) and (47), the measurand interval in this case is of the form

$$\vec{x} = [1.99979 - 0.00030 \; ; \; 1.99979 + 0.00030] = $$
$$= [1.99949 \; ; \; 2.00009]\,\text{mV}. \quad (48)$$

DETERMINING THE UNCERTAINTY OF THE OUTPUT MEASURAND

Knowing the measurand interval allows you to determine the value of uncertainty defined as the radius of this interval [14]. Denoting the limits of the uncertainty interval as $\underline{x}$ and $\overline{x}$ respectively, the uncertainty u is defined as:

$$u = \text{rad}(\vec{x}) = \frac{\overline{x} - \underline{x}}{2} \quad (49)$$

For a confidence level of $p = 0.95$, the uncertainty is called expanded and denoted by U [3]. According to expressions (46) and (49), there is

$$u = \frac{\overline{x} - \underline{x}}{2} = \frac{\hat{x} + \overline{u} - (\hat{x} + \underline{u})}{2} = \frac{\overline{u} - \underline{u}}{2} \quad (50)$$

which means that the uncertainty can also be defined as the radius of the uncertainty interval.

Example 6. For the measurand interval in the form (48), the uncertainty at $p = 0.95$ is

$$U = \text{rad}(\vec{x}) = \frac{2.00009 - 1.99979}{2} = $$
$$= 0.0003\,\text{V} = 0.3\,\text{mV} \quad (51)$$

The same value as (51) is obtained from the limits of the uncertainty interval. According to the expression (50) and the values obtained from the histogram in Figure 3, the expanded uncertainty takes the value

$$U = \text{rad}(\vec{x}) = \frac{0.3 - (-0.3)}{2} = 0.3\,\text{mV} \quad (52)$$

The output uncertainty makes it possible to characterize the metrological properties of an algorithm by a single number, and is therefore useful for comparing the accuracy of algorithms performing, for example, under different measurement conditions. On the other hand, for the purpose of detailed analysis of the factors affecting the inaccuracy of an algorithm, it is necessary to describe the sources of errors and determine their interrelationships. This kind of information is provided by the algorithm's output error budget, which is a summary of these errors with a description of their distributions and standard deviations.

The errors in the output of the exemplary algorithm covered by Experiment 2 are presented in Table 1. The standard deviation of the quantization error is given by expression (17). This error is of normal distribution, as the conditions of the Central Limit Theorem [3] are satisfied in this case, because the propagation of the error through the algorithm. The noise error at the input of the algorithm has a standard deviation of 1 mV, and at the output it has a standard deviation of 0.1 mV, since the random error propagation coefficient according to expression (16) is 0.1. The temperature error, which is the sum of the zero error and the slope error, is of the rectangular distribution and has the standard deviation value described by relation (33).

Based on the error Equation 7 of the algorithm, it is possible to determine the distribution of the total error for known partial error distributions using a convolution of probability density functions, as long as these errors are not correlated. However, performing convolution is a complex problem, especially when some of

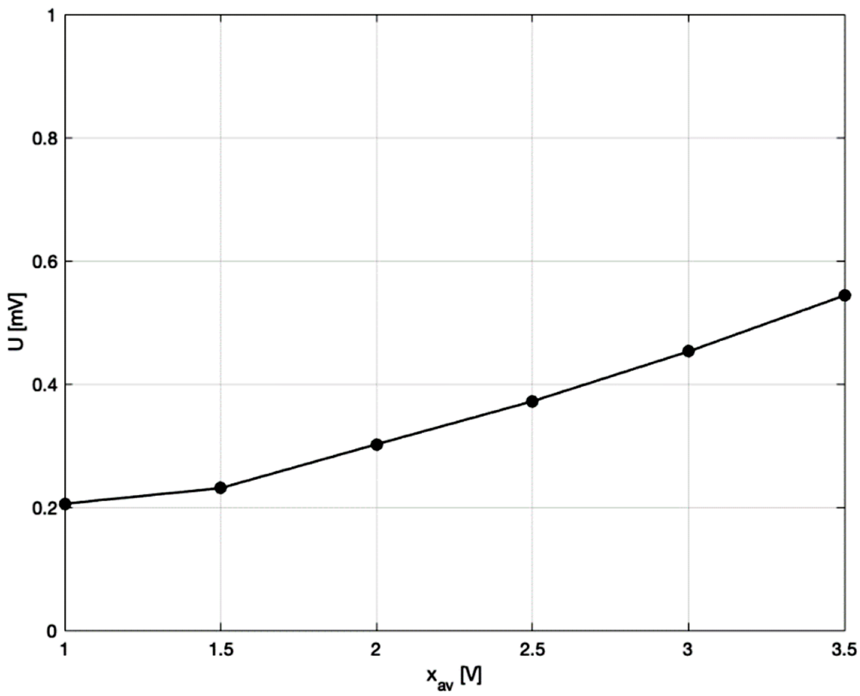

Figure 4. Dependence of the uncertainty of the exemplary algorithm on the output measurand being the average value of the input voltage waveform

Table 1. Error budget at the output of the exemplary algorithm

Errors caused by	Quantization (Type A)	Noise (Type A)	Temperature (Type B)
Distribution type	Normal	Normal	Rectangular
Standard deviation [mV]	$\sigma_{q,out}$= 0.029	$\sigma_{noi,out}$ = 0.100	$\sigma_{temp,out}$ = 0.115

the distributions are determined experimentally. Therefore, the most efficient way to determine the distribution of the total error is to perform a simulation experiment using the Monte Carlo method, as shown in Experiment 2. An experiment of this kind uses complex model composed of the measurement description and numerical processing, the correctness of which requires verification. For this purpose, the algorithm's error equation can be used to calculate the standard deviation of the total error based on knowledge of the standard deviations of the partial errors. The comparison of the standard deviation obtained by this method with the standard deviation of the total error calculated on the basis of the set of error values obtained as a result of the simulation experiment makes it possible to conclude that the model used in the Monte Carlo experiment is correct.

Example 7. The output error of the exemplary algorithm is the sum of partial errors presented in Table 1. These errors are uncorrelated, so the standard deviation of the output error is described by the expression [2]

$$\sigma_{out} = \sqrt{\sigma^2_{q\ out} + \sigma^2_{noi,out} + \sigma^2_{temp,out}} \quad (53)$$

in which $\sigma_{q,out}$, $\sigma_{noi,out}$ and $\sigma_{noi,out}$ are the standard deviations of the output errors caused by quantization, noise and temperature. When the numerical values from this table are introduced into expression (53), one obtains

$$\sigma_{out} = \sqrt{0.029^2 + 0.1^2 + 0.115^2} = 0.156 \text{ mV} \quad (54)$$

that is, the same value as the result of Experiment 2. This means that the complex processing model used in this experiment is correct.

Verification of the complex processing model allows it to be used in simulation experiments aimed at determining the distributions of partial errors, analyzing the correlations between them and comparing their values in order to take measures to reduce the share of large errors in the total error. This model can also be used to determine the dependence of uncertainty as a function of parameters that influence the processing. For the exemplary algorithm, the model was used to calculate the uncertainty depending on the value of the output measurand. The obtained results are presented in Figure 4.

Intermediate values of uncertainty between nodes determined by simulation can be calculated using segmental linear approximations of the relationship shown in Figure 4 [2]. This way is numerically simple and sufficiently accurate when considering the requirements for uncertainty values. If the dependencies are strongly nonlinear, the number of nodes can be increased according to achieve the required accuracy of the uncertainty calculation.

CONCLUSIONS

The mathematical framework described was significantly expanded by introducing a modified concept of the measurand and a mathematical model of the measurand. This allowed the algorithm to be decomposed into a processing equation and an error equation, which in turn allowed for the classification of errors into type A and type B. The division of measurement errors into these two categories is particularly important when analyzing the errors of an algorithm for processing measurement data, because they propagate by the algorithm in different ways. Analysis of the error propagation from the input to the output of the algorithm makes it possible to determine the propagation coefficients, which are different for random errors referred to as type A and errors of type B with constant values in the measurement window. Type A errors can be filtered by the algorithm, as is the case with the algorithm under consideration, or amplified, as is the case with dynamic reconstruction algorithms. On the other hand, additive errors of type B are transferred from the input to the output of the algorithm with a constant coefficient mostly equal to 1. In the case of multiplicative errors of type B, the transfer coefficient depends on the value of the output measurand of the algorithm. Both types of errors are described at the output in probabilistic categories allowing determination of the distribution of the total error, which is the basis for calculating the uncertainty of the output measurand.

The mathematical apparatus presented here can be applied to calculate uncertainty in measurement systems used both in laboratories and in industry conditions. This is related to the possibility of using it in complex measurement conditions occurring in industry, where nonlinear sensors operating in dynamic conditions are used, and measurement data sequences are processed by linear and nonlinear algorithms. Such a wide application of the described method is possible due to the simplicity of the obtained procedures for determining uncertainty. The importance of this simplicity is visible when comparing the determination of uncertainty for the same type of nonlinear algorithms. The division of errors into categories A and B allows for the analysis of error propagation from input to output of both types of the processing algorithms, which makes it possible to determine impact of errors on the uncertainty of the output result and, therefore, to use measures to reduce its inaccuracy.

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