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Balanced fuzzy negation: Theory, generators and motivations

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ABSTRACT

Balanced fuzzy sets (BFS), introduced in 2006, offer a new perspective on describing reality. In fuzzy logic, the negation of partial membership becomes another form of partial membership, which leads to the indistinguishability of a set from its complement. In BFS, where the sign of a value determines the nature of membership – positive or negative – this ambiguity is removed, and negation regains its semantic role as an opposite. This is consistent with the natural interpretation of human reasoning, in which the strengths of positive and negative premises are treated as equally important. It allows for the modelling of positive, negative, and neutral information. Moreover, they also account for the fact that the shape of the membership and non-membership functions can change. The innovation of the presented work lies in its new approach to negation, which takes into account the need to generate the negations adapted to the observed processes representing human perception of reality. This allows understanding that the negation of a value symbolizing partial excess is a partial deficiency, meaning that the negation of values representing the information that something is partially positive becomes the fact that something is partially negative. In other words, the negation in BFS reflects the transition from positive information to its opposite, in a manner observed in natural processes. This negation determines how fuzzy membership in a given set translates into the uncertainty of membership in its complement. Balanced fuzzy negation was presented, demonstrating the construction of BNF from classical negations and generating functions. Its various properties and the differences between supplement and complement in BFS were discussed. Fundamental relationships with operators in classical fuzzy sets were presented. Properties and examples (including Yager, power, and trigonometric operators) were provided. The results clarify the concepts of negation and complement in BFS and suggest directions for further research on balanced fuzzy operations in [-1,1]. This enables more informed decision-making, risk analysis, sentiment assessment, and other engineering tasks that require considering positive and negative ratings, as well as neutral or no information.

Keywords: balanced fuzzy sets, fuzzy negations, fuzzy operators, fuzzy logic, mathematical modelling.

INTRODUCTION

Classical set theory is deeply rooted in well-defined mathematical concepts. However, modelling processes occurring in society or describing human decision-making is practically impossible using Boolean logic. Łukasiewicz's introduction of three-valued logic was a key advance, opening the possibility of a better description of observed reasoning.

It aimed to solve the problem of statements the truth value of which cannot yet be determined, such as statements about future events [1]. It is worth emphasizing that this approach is currently used in the programming branches of computer science, including through the use of values such as NULL, NILL, or "undefined."

Later, fuzzy sets, introduced by Zadeh in 1965 [2], proved to be an effective way to describe uncertain and ambiguous reality. However, the interpretation of negation in fuzzy logic remains ambiguous, at least not in all applications. Does complementation truly indicate falsity or uncertainty? Modern balanced fuzzy sets extend this approach by introducing a separate non-membership function [3]. In this model, the values (0, 1) denote membership, [-1, 0) non-membership, and 0 denotes neutrality or lack of information. This distinction is particularly important in domains where positive and negative evaluations coexist, for example, in medical or psychological diagnostics, financial risk assessment, and sentiment analysis.

Example. Consider a set A representing individuals with a very high IQ. Membership can be modelled as μ_{hIQ} in fuzzy sets and as η_{hIQ} in balanced fuzzy sets:

$$\mu_{hIQ}(x) = \begin{cases} 0 & x < 110 \\ 0.01x - 1.1 & x \in [110,120) \\ 0.1 \cdot \left(10^{\frac{x}{120}} - 9\right) & x \in [120,130) \\ \min(0.009x - 0.85, 1) & x \geq 130 \end{cases}$$

$$\eta_{hIQ}(x) = \begin{cases} \min(-1, 0.0004x^2 - 3.24) & x < 90 \\ 0 & x \in [90,110) \\ 0.01x - 1.1 & x \in [110,120) \\ 0.1 \cdot \left(10^{\frac{x}{120}} - 9\right) & x \in [120,130) \\ \min(0.009x - 0.85, 1) & x \geq 130 \end{cases}$$

where: x is the IQ score.

In summary, balanced fuzzy sets allow for better modelling and interpretation of real-world processes, encompassing concepts such as excess and deficiency, satisfaction and dissatisfaction, as well as presence and absence. They also play an important role in describing the processes by which people evaluate observed states.

The article is divided into the following order: first, the basic concepts of classical, fuzzy, and balanced sets were reviewed, highlighting their structural differences and the motivations for their extension. The concept of balanced fuzzy negation was then introduced, along with its formal definition, properties, and illustrative examples. Methods for generating balanced fuzzy negations from classical negations were then presented. Finally, the concepts of complement and complementation in balanced fuzzy sets were examined, followed by conclusions and directions for further research.

OVERVIEW OF THE BASIC CONCEPT OF SETS

This article presents some insights into the fundamental principles of balanced fuzzy sets. A crisp set membership is defined by a characteristic function, which assigns each element in the universal set X either 1 (indicating the element is in the set A) or 0 (if not in A).

In the construction of standard fuzzy sets, fundamental set operations such as union, product, and complement are defined as:

$$f_{A\cup B}(x) = max\left(f_A(x), f_B(x)\right), f_{A\cap B}(x) =$$
$$= min\left(f_A(x), f_B(x)\right), f_A^C(x) = 1 - f_A(x).$$

However, it should be added that while crisp sets provide only a binary form of membership, fuzzy sets allow the use of values from the interval [0,1] [2]. A fuzzy set A is defined by a membership function $\mu_A : X \to [0,1]$, which assigns to each element its membership degree. On the basis of the properties of crisp sets, operations for fuzzy sets are constructed. The intersection and union of fuzzy sets are defined as follows (compare with [4] and [5]):

$$\mu_{A\cup B}(x) = S\left(\mu_A(x), \mu_B(x)\right), \mu_{A\cap B}(x) = T\left(\mu_A(x), \mu_B(x)\right),$$

where: S is a t-conorm and T is a t-norm.

Fuzzy negation is defined as a function $n : [0,1] \to [0,1]$ that satisfies the following conditions: boundary conditions ($n(0) = 1$ and $n(1) = 0$ and monotonicity ($n(x) \leq n(y)$ for $x \geq y$). Moreover, strict negations also require involution ($n(n(x)) = x$).

However, the classical fuzzy set operations do not fully consider all aspects of uncertainty. For this reason, balanced fuzzy sets extend fuzzy sets by introducing both membership and non-membership degrees [3]. The characteristic function for a balanced fuzzy set A is defined as:

$$\eta_A(x) = \begin{cases} \mu_A^1(x) & x \text{ is in A} \\ 0 & \text{is neutral} \\ \mu_A^2(x) & x \text{ is not in A} \end{cases},$$

where: $\mu_A^1(x) \in (0,1]$ and $\mu_A^2(x) \in [-1,0)$ (see [6]).

This distinction between positive and negative membership allows for a better representation of uncertainty than standard fuzzy sets. Operations on balanced fuzzy sets are defined using balanced t-conorms SB and t-norms TB (see [6]):

$$\eta_{A\cup B}(x) = SB\left(\eta_A(x), \eta_B(x)\right), \eta_{A\cap B}(x) = TB\left(\eta_A(x), \eta_B(x)\right).$$

As was presented in [7] in these operations can be generated by using nullnorms and uninorms.

Theorem 1. ([7], Theorem 4) If U is a representable uninorm with the additive generator

$$u(x) = \begin{cases} -t(2x) & x \in [0, e] \\ s(2x-1) = t(2-2x) & x \in (e, 1] \end{cases},$$

where: $t\colon[0,1]\to[0,+\infty]$ and $s(x) = t(1-x)$ are additive generators of t-norm T, and is dual t-conorm S (respectively), and $e = 0.5$, then operations $SB_r\colon[-1,1]^2\to[-1,1]$ defined by

$$SB_r(x,y) = \begin{cases} r & (x,y) \in \{(-1,1),(1,-1)\} \\ f^{-1}\left(u^{-1}\left(u(f(x)) + u(f(y))\right)\right) & \text{otherwise} \end{cases},$$

are balanced t-conorms and $r \in \{-1,1\}$.

Theorem 2. ([7], Theorem 6) If V is a nullnorm on [0,1] with zero element $z \in [0,1]$ and dual operations T and S, then $TB\colon[-1,1]^2\to[-1,1]$ defined by

$$TB(x,y) = \begin{cases} S(x+1, y+1) - 1 & x,y \in [-1,0] \\ T(x,y) & x,y \in [0,1] \\ 0 & \text{others} \end{cases}.$$

Example. Let us demonstrate the use of these logical connectives in a Petri Net (PN). Recall that PN is a mathematical model for describing processes in which various events occur in parallel, in a specific order, or depending on conditions. It is used to simulate industrial processes or information flow. Let us use the example of the rules presented in the work [8] using the balanced operators SB_{TanH} and TB_{TanH} [6, 7].
Let us see the following rule:

$$IF[(d_1 = 0.8\ AND\ d_2 = 0.5)\ OR\ d_3 = -0.6]\ THEN d_4$$
$$(CF = 0.9),$$

given three input states and calculating the expected output state, with a given certainty factor to the rule CF (Figure 1).

The example shows how the knowledge about resources or their lack (with a given rule) will affect the final result. The implementation of this rule shows that the result will be some shortage.

To sum up, owing to such operations, it is possible to integrate multiple sources of information, e.g. sensors in an autonomous system, where some data are positive, others negative. In this sense, as an example of the application of these operations, the autonomous car system combines the data from cameras (positive – visibility signals) and radars (negative – threat signals) to calculate a balanced safety state.

Another, but equally interesting, issue is the modelling of decision-making processes where some options are partially exclusive and others are neutral. This situation is observed in a recommendation system that combines user opinions – positive, negative, and neutral – to determine the final product rating.

However, it should be noted, that in order to generate rules corresponding to real processes, negation is still needed.

MOTIVATIONS

In artificial intelligence, and especially in neural networks, activation functions that return only positive values are used, but also those that can take on negative values. This allows the neuron to respond both positively and negatively to the input signal - allowing the network to better distinguish between positive and negative influences. Examples of such functions include tanh, Leaky ReLU, and ELU (see [9, 10]). This allows the network not only to amplify signals, but also to attenuate or "invert" certain aspects of information. Therefore, one might consider using the concept of balanced fuzzy negation in the design

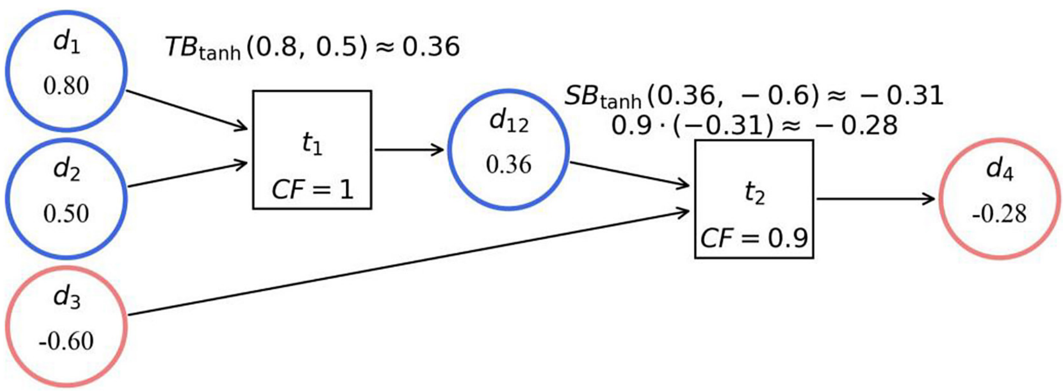

Figure 1. An example of rule in balanced fuzzy Petri Net using balanced operators

of such processes, so that the model can account for the gradual rejection or inversion of the signal.

However, in order to justify that the BFS consideration has a mathematical justification, let us start with the discussion of set complement and review one of aspect of classical crisp set theory, i.e., the "Axiom of Extensionality" [11]:

"Two sets are equal (i.e., the same set) if they contain exactly the same elements. This states that sets are defined by their elements". Extensional characterisation using characteristic functions with values {0,1} is not transferable to fuzzy sets; that is, with partial membership, a set and its complement do not provide an exclusive-exhaustive classification of the elements.

Example. Let us consider a space in which all elements belong to a set with membership degree 0.5 – applying classical fuzzy negation, defined as $n(x) = 1 - x$, we obtain a situation in which the set and its complement are indistinguishable. The concept of complement, therefore, differs from the classical notion, in which an element must belong to a set or its complement. This highlights the fundamental distinction between complement in classical set theory and fuzzy complement. The lack of a clear distinction between membership and non-membership in fuzzy sets challenges traditional assumptions and requires a reconsideration of the basic principles of such set representation.

In classical fuzzy logic, negation changes the degree of truth but does not lead to a qualitative change in interpretation. In other words, the negation of an element with partial membership merely becomes another form of the same partial membership. As a result, the set and its complement differ only in value, not in the direction (sign) of information (i.e.: partial positive stay partial positive). Therefore, low membership and high non-membership can be represented in essentially the same way, i.e., as numbers close to zero. This blurs the intuitive meaning of negation as an opposite.

In BFS, this problem is eliminated by introducing a bipolar representation of information. The sign of the value has a specific meaning; that is, the positive part describes positive arguments "for" and the negative part describes arguments "against." Negation is therefore not simply a scaling inversion, but a transformation that changes the nature of the information. In summary, what was given as positive becomes negative, and vice versa. This makes BFN a true opposite, consistent with the common understanding of the principle "A becomes not A."

The balanced fuzzy supplement operator and the balanced fuzzy negation of the set from Figure 2 are presented in Figures 3a and 3b. They illustrate that the supplement operator reflects various forms of uncertainty about the full membership or non-membership of an element in the set. The negation operator, on the other hand, illustrates the process of negating membership – defined on the basis of a given, appropriately selected form of negation appropriate to the phenomenon being analysed.

It should be emphasised that the concepts of BFS and BFN are relatively new. Although the observation that negative information must also be represented and analysed also appears in intuitionistic fuzzy sets (IFS) [12]. There, information is expressed in a dual manner (containing both positive and negative information

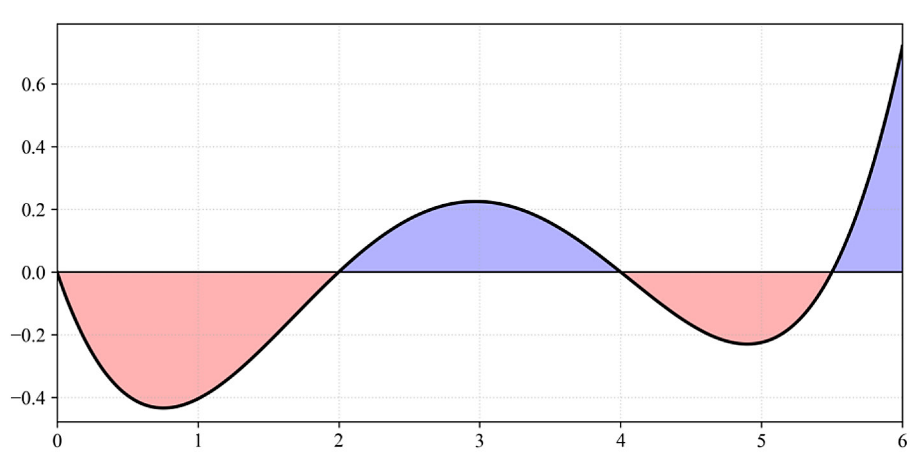

Figure 2. Figure of a sample balanced fuzzy set A

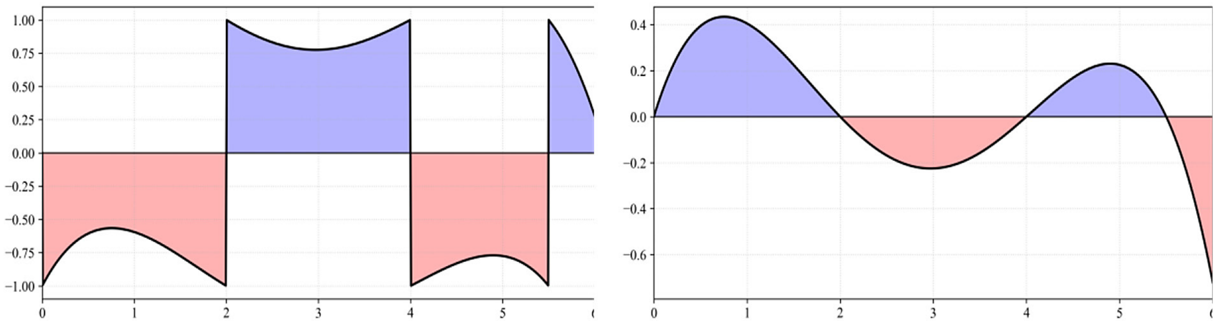

Figure 3. Figures of a samples a) balanced fuzzy supplements and b) balanced fuzzy negation of *A*

simultaneously). Table 1 provides a concise summary of the relational properties in all three sets.

The BFS approach is much closer to natural human reasoning, in which positive and negative factors are treated as two complementary but independent sources of information [13] Psychological research on decision-making shows that people interpret negation not as a weakened confirmation, but as the emergence of a qualitatively different premise.

Example. Economic research [13] shows that people react to losses and gains as two different types of information. A loss is not interpreted as a "negative gain," but as a clear negative signal. This behaviour is closely related to the bipolarity of the BFS, in which positive and negative values represent two different types of arguments ("for" and "against"). Thus, BFN reflects a psychological mechanism that is well documented empirically.

To sum up, BFS restores negation to a meaning that is both mathematically unambiguous and consistent with the intuitive logic of situational judgment, which is bipolar in nature.

Table 1. Main comparison of negations of fuzzy sets, Atanassov IFS and BFS

Properties of set with proper negations	Fuzzy sets	Intuitionistic fuzzy sets	Balanced fuzzy sets
Meaning	Numeric kind of inversion. Negation does not represent true opposition in common sense.	Switches membership and non-membership, in 2D. Suitable for data that has a dual form.	True semantic opposition.
Representation of negative information	None (the whole negative information is in 0).	Explicit via $\nu(x)$.	Directly encoded as $\eta(x) < 0$.
Uncertainty	Not modelled.	$\pi(x)$ remains unchanged.	Values near 0.

BALANCED FUZZY NEGATIONS

The fuzzy negation can be interpreted as quantifying the extent to which an element does not belong fully to a set. In this sense, fuzzy negation is a method for computing the supplementary or deficit value required to reach complete membership. Extending this concept to balanced fuzzy sets, the corresponding operator was introduced. The supplement operator in balanced fuzzy sets, denoted as I, works analogously to fuzzy negation concept. Specifically:

- For values in the range $(0,1]$, it computes the supplement of full membership in the set.
- For values in the range $[-1,0)$, it determines the supplement to full non-membership.

Therefore, to formalise the fuzzy balanced supplementary operator, let us first recall the reversal operation $N(x) = -x$ (see [3]), so

$$I(x) = \begin{cases} \mathrm{n}(\mathrm{x}) & x > 0 \\ 0 & x = 0 \\ \mathrm{N}\left(\mathrm{n}\left(\mathrm{N}(\mathrm{x})\right)\right) & x < 0 \end{cases}, \tag{1}$$

where n is a fuzzy negation.

On the basis of Eq. (1) the operator I is odd. Hence, it retains the properties known for fuzzy negation, on each interval: $[0,1]$ and $[-1,0]$.

Some properties:

- I is a non-increasing function
- if n is continuous, then I is continuous
- if n is an involution, then I is an involution

in each interval $(0,1)$ and $(-1,0)$, but not on the whole interval $[-1,1]$. Let us note that $n|_{(0,1)} = I_{(0,1)}$, but $n \neq I|_{[0,1]}$, because $n(0) = 1$, and $I(0) = 0$.

Example. Let us present some examples of supplementary operators generated directly from fuzzy negation (see Tab. 2 and Fig. 4a)- Fig. 4c)).

Table 2. The example of balanced fuzzy supplements

Name of supplementary operator	Fuzzy negation	Supplementary operator
Classical supplement	$n(x) = 1 - x$	$I(x) = \begin{cases} 1-x & x>0 \\ -1-x & x<0 \\ 0 & x=0 \end{cases}$
Even power supplement	$n(x) = 1 - x^p$	$I(x) = \begin{cases} 1-x^p & x>0 \\ -1+x^p & x<0 \\ 0 & x=0 \end{cases}$ p – even number
Odd power supplement	$n(x) = 1 - x^p$	$I(x) = \begin{cases} 1-x & x>0 \\ -1-x^p & x<0 \\ 0 & x=0 \end{cases}$ p – odd number
Square root supplement	$n(x) = 1 - \sqrt{x}$	$I(x) = \begin{cases} 1-\sqrt{x} & x>0 \\ -1+\sqrt{x} & x<0 \\ 0 & x=0 \end{cases}$
Cosine supplement	$n(x) = \frac{1}{2} \cdot (1 + \cos(\pi x))$	$I(x) = \begin{cases} \frac{1}{2} \cdot (1+\cos(\pi x)) & x>0 \\ -\frac{1}{2} \cdot (1+\cos(\pi x)) & x<0 \\ 0 & x=0 \end{cases}$
Sugeno supplement	$n(x) = \frac{1-x}{1+\lambda x}$	$I(x) = \begin{cases} \frac{1-x}{1+\lambda x} & x>0 \\ \frac{-1-x}{1-\lambda x} & x<0 \\ 0 & x=0 \end{cases}, \lambda > -1$

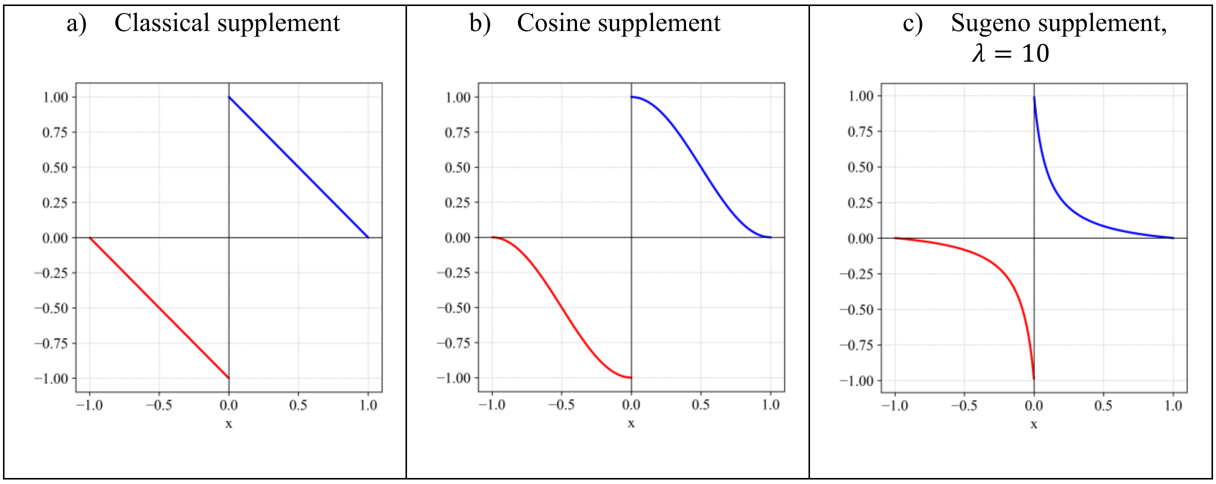

Figure 4. Figures of balanced fuzzy supplements

Definition 1. [[6], Definition IV.3.] A decreasing function $NI: [-1,1] \to [-1,1]$ that satisfies the following conditions: $NI(0) = 0$, $x \cdot NI(x) \leq 0$, for all $x \in [-1,1]$ is called a balanced fuzzy negation (BFN). NI is a strict balanced fuzzy negation when it is a bijection, a strong negation when NI is an involution. When NI is an involution on $(-1, 0) \cup (0,1)$ then NI is called a narrowed involution.

Corollary. The balanced negation NI has exactly one equilibrium, namely $x = 0$.

Corollary. If $n: [0,1] \to [0,1]$ is fuzzy negation then, fuzzy balanced negation can be generated based on:

$$NI_I = \begin{cases} n(x) - 1, & x \in (0,1] \\ 1 - n(-x), & x \in [-1,0) \\ 0, & x = 0 \end{cases} \quad (2)$$

Example. It should be emphasised that it is not sufficient to change the sign before the negation of n in the interval $[0, 1]$, because the resulting function is not decreasing:

$$neq_n(x) = \begin{cases} -n(x), & x \in (0,1], \\ n(-x), & x \in [-1,0). \\ 0 & 0 \end{cases}$$

Example. Using the example of fuzzy Yager negation $n(x) = \left(1 - x^\lambda\right)^{1/\lambda}$, the corresponding supplementary operator I and balanced fuzzy negation NI can be seen (see Fig. 5a) – 5b)).

a) b)

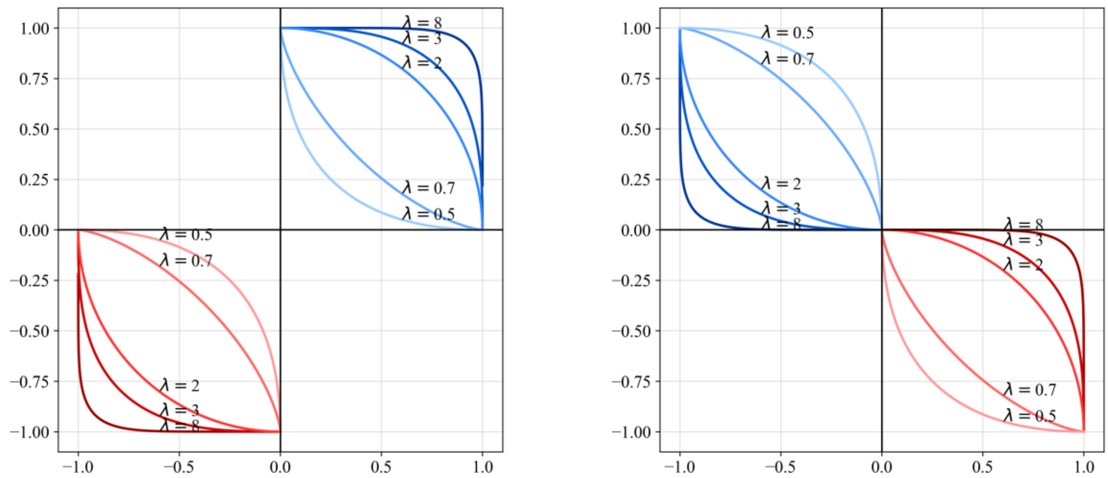

Figure 5. a) Supplementary Yager operators I and b) Yager balanced fuzzy negation NI

$$I(x) = \begin{cases} \left(1 - x^\lambda\right)^{\frac{1}{\lambda}} & x \in (0,1] \\ -\left(1 - (-x)^\lambda\right)^{\frac{1}{\lambda}} & x \in [-1,0) \\ 0 & x = 0 \end{cases}, \quad NI(x) = \begin{cases} -1 + \left(1 - x^\lambda\right)^{\frac{1}{\lambda}} & x \in (0,1] \\ 1 - \left(1 - (-x)^\lambda\right)^{\frac{1}{\lambda}} & x \in [-1,0) \\ 0 & x = 0 \end{cases}.$$

However, neq_n is increasing on both intervals and has the following form (see Fig. 6):

$$neq_{n(x)} = \begin{cases} -\left(1 - x^\lambda\right)^{\frac{1}{\lambda}} & x \in (0,1] \\ \left(1 - (-x)^\lambda\right)^{\frac{1}{\lambda}} & x \in [-1,0) \\ 0 & x = 0 \end{cases},$$

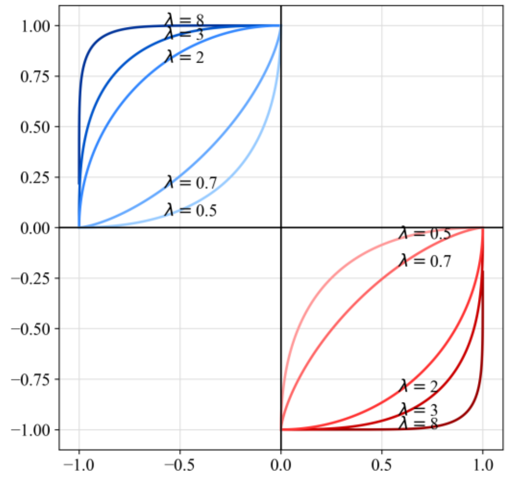

Figure 6. Operation neq_n – example of not balanced fuzzy negation

Corollary. The simplest extension of the left part of rule Eq. (2) to obtain a fuzzy balanced negation is to change the sign of the function I, e.i.: $NI_I(x) = I(N(x)) = N(I(x))$.

Extending fuzzy negations to balanced domains ensures that neutral values (0) remain stable reference points. For example, in a patient health monitoring system, a balanced assessment allows for a clear distinction between a patient with normal vitality (0) and those with improvement or deterioration, while maintaining interpretive neutrality.

Humans interpret gradual changes, inconsistencies, and uncertainty. Their practical counterparts emerge in AI-based decision-making systems, which require symmetrical and evaluative responses to positive and negative factors. This also corresponds to human perception of gradually changing sentiments. For example, in opinion mining, when a post is re-evaluated using multiple models (context, tone, sarcasm), the final result remains directionally consistent – negativity does not randomly change to positivity through repeated processing. Certain properties that allow for modelling negation generation processes follow from the following propositions:

Proposition 1. Let n_1, n_2: $[0,1] \to [0,1]$ be the fuzzy negations, then

$$NI(x) = \begin{cases} n_1(x) - 1 & x \in (0,1] \\ 1 - n_2(-x) & x \in [-1,0) \\ 0 & x = 0 \end{cases}$$

is the balanced fuzzy negation.

Proposition 2. Let $f: [-1,1] \to [-1,1]$ be a strictly decreasing bijection with $f(1) = -1, f(0) = 0$ and $f(-1) = 1$. Then $NI = f|_{[-1,1]}$ is a balanced fuzzy negation.

Proposition 3. Let $f: [-1,1] \to [-1,1]$ be a strictly decreasing bijection with $f(1) = -\lambda$, $f(0) = 0$ and $f(-1) = \lambda$. Then $NI = \frac{1}{\lambda} f|_{[-1,1]}$, where $\lambda \in \mathbb{R} - \{0\}$ is a balanced fuzzy negation.

Proposition 4. Let NI be the bijection. If NI is a balanced fuzzy negation, then NI^{-1} is balanced fuzzy negation.

Proposition 5. Let $k \in \mathbb{N}$, $k \geq 3$. If NI_k are balanced fuzzy negations, then the composition of an odd number of balanced fuzzy negations is balanced negations.

Example. To illustrate Proposition 5, let us consider negations

$$NI_1(x) = \begin{cases} -1+\sqrt{1-x} & x \in [0,1] \\ 1-\sqrt{1+x} & x \in [-1,0] \end{cases},$$
$$NI_2(x) = -\sin\left(\pi \cdot \frac{x}{2} + 2t\cdot\pi\right), NI_3(x) = -x^{2\cdot s+1}.$$

Thus, we have:

$$NI_1 \circ NI_2^{-1} \circ NI_3(x) = \begin{cases} -1+\sqrt{1-\frac{2}{\pi}\arcsin(x^{2s+1})} & x \in [0,1] \\ 1-\sqrt{1-\frac{2}{\pi}\arcsin(x^{2s+1})} & x \in [-1,0] \end{cases}.$$

Of course, multiple composition of the same negation is also negation

$$(NI_1 \circ NI_1 \circ NI_1)(x) = \begin{cases} -1+\sqrt[8]{1-x} & x \in [0,1] \\ 1-\sqrt[8]{1+x} & x \in [-1,0] \end{cases}.$$

GENERATION OF BALANCED FUZZY NEGATIONS

Following the properties of the function n, the following results are obtained.

Theorem 3. If fuzzy negation n is a function, which can be extended to even function $n: [-1,1] \to [0,1]$, then can be generated as

$$NI(x) = \begin{cases} -1+n(x) & x \geq 0 \\ 1-n(x) & x < 0 \end{cases}.$$

Moreover, if n is a strict fuzzy negation, then N is a strict balanced fuzzy negation.

Proof. Since $n(x) = n(-x) \in [0,1]$, so we have $-1+n(x) \in [-1,0]$, $1-n(x) \in [0,1]$ and $NI(0) = -1+n(0) = -1+1 = 0$. Moreover, if n is a strict fuzzy negation, then because of extension of $n: [-1,1] \to [0,1]$ we obtain $NI(-1) = 1-n(-1) = 1-n(1) = 1$ and $NI(1) = n(1)-1 = 0-1 = -1$. Because n is decreasing function in $[0,1]$, then $N(x) = n(x)-1$ is decreasing. Also, extended even function n in $[-1,0]$ is increasing, so $N(x) = 1-n(x)$ is decreasing.

Note that Theorem 3 applies, among other things, to the analysis of emotional signals, where reactions can be both positive and negative, and various events can cause mood reversals. Therefore, it is useful in sentiment analysis, for example, in social media. There, the system interprets the tone of a statement, assigning a positive value to positive emotions and a negative value to negative emotions. The situation is somewhat different in diagnostic systems that analyse both positive and negative symptoms. Here, for example, a medical system considers both signs of health (positive) and symptoms of disease (negative), thus generating a balanced picture of the patient's condition.

Theorem 4. Let an odd function $f: [0,1] \to [0,1]$ be given. If the fuzzy negation $n: [0,1] \to [0,1]$ is a function of the form $n(x) = 1-f(x)$, then $NI(x) = -f(x)$ is a balanced fuzzy negation. Moreover, if n is a strict fuzzy negation, then NI is a strict balanced fuzzy negation.

Proof. Because $n(x) \in [0,1]$, we have that $1-f(x) \in [0,1]$, so $f(x) \in [0,1]$ for all $x \in [0,1]$. From assumption that f is an odd function we know, that $f(-x) \in [-1,0]$ for all $x \in [0,1]$. Since $N(x) = n(x)-1 = -f(x)$ and translation does not change monotonicity so N is a decreasing function. Moreover, because f is an odd function and because n is fuzzy negation, which give us that $n(1) = 0$ and $n(0) = 1$, then $NI(1) = -1$, $NI(-1) = 1$ and $NI(0) = 0$.

However, not all negations operate the same way. In the case of modelling adaptive behaviour, where a negative response is likely to occur, a negation that is the proportional opposite of a positive response is more likely to be used.

Theorem 5. Let an even function $f: [0,1] \to [0,1]$ be given. If the fuzzy negation $n: [0,1] \to [0,1]$ is a function of the form n(x)=$1 - f(x)$, then

$$NI(x) = \begin{cases} -f(x) & x \geq 0 \\ f(x) & x < 0 \end{cases}$$

is a balanced fuzzy negation. Moreover, if n is a strict fuzzy negation, then N in a strict balanced fuzzy negation.

Proposition 6. If domain of the function f is the set of $x \geq 0$ and the fuzzy negation $n: [0,1] \to [0,1]$ is a function of the form $n(x) = 1 - f(x)$, then

$$NI(x) = \begin{cases} -\mathrm{f(x)} & \mathrm{x} \geq 0 \\ \mathrm{f(-x)} & \mathrm{x} < 0 \end{cases}$$

is a balanced fuzzy negation. Moreover, if n is a strict fuzzy negation, then N in a strict balanced fuzzy negation.

Example. Similarly, as we have given the basic families of fuzzy balanced supplements, let us give the corresponding fuzzy balanced negations (see Tab. 3 and Fig. 7a) – 7c))

Table 3. The example of fuzzy balanced negation

Name of BFN	Name of supplementary operator	Supplementary operator
Reversal operator	classical supplement	$N(x) = NI(x) = -x$
Even power negation	even power supplement	$NI(x) = \begin{cases} -x^p & x > 0 \\ x^p & x < 0 \end{cases}$ p – even number
Odd power negation	odd power supplement	$NI(x) = -x^p$ p – odd number
Square root negation	square root supplement	$\mathrm{I}(x) = \begin{cases} 1 - \sqrt{x} & x > 0 \\ -1 + \sqrt{x} & x < 0 \end{cases}$
Cosine negation	cosine supplement	$NI(x) = \begin{cases} \frac{1}{2} \cdot (-1 + \cos(\pi x)) & x > 0 \\ \frac{1}{2} \cdot (1 - \cos(\pi x)) & x < 0 \\ 0 & x = 0 \end{cases}$
Sugeno negation	Sugeno supplement	$NI(x) = \begin{cases} \frac{1-x}{1+\lambda x} - 1 & x > 0 \\ 1 - \frac{1+x}{1-\lambda x} & x < 0 \end{cases}$, $\lambda > -1$

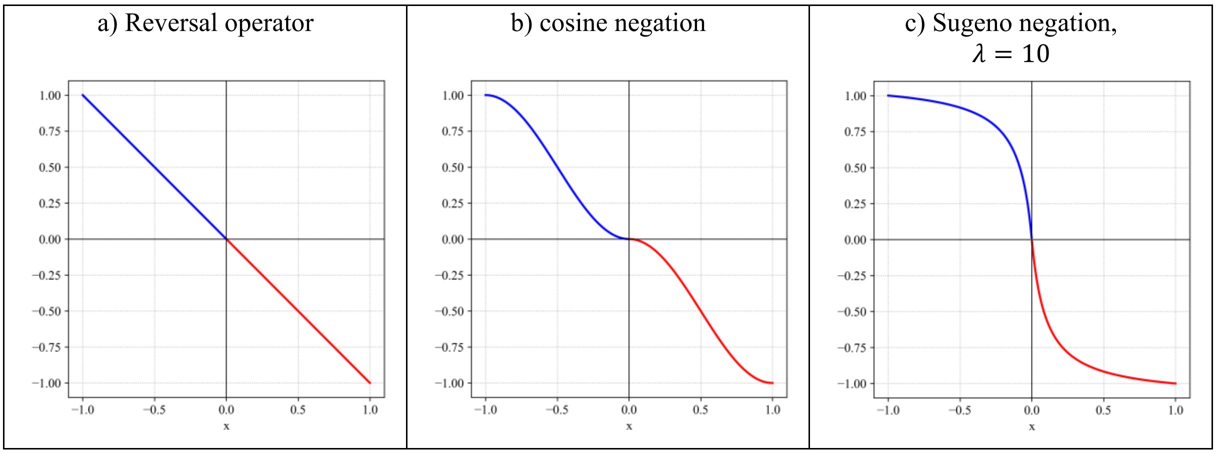

Figure 7. Figures of balanced fuzzy negation

On the basis of these considerations, it is generalised as follows:

Theorem 6. If there exists a strictly increasing and continuous function $g: [-1,1] \to \mathbb{R}$ such that $g(-1) + g(1) = 2g(0)$, then

$$NI(x) = g^{-1}\big(g(1) + g(-1) - g(x)\big), for\ all\ x \in [-1,1] \quad (3)$$

NI is a balanced fuzzy negation.

Proof. Firstly, we prove that NI is a balanced fuzzy negation. We need to check that it is a decreasing function. From assumption, that g is increasing function, for $x_1, x_2 \in [-1, 1]$ and $x_1 \le x_2$ we get that function $f(x) = g(1) + g(-1) - g(x)$ is decreasing function, i.e.:

$$g(1) + g(-1) - g(x_1) \le g(1) + g(-1) - g(x_2).$$

Now, based on fact that the inverse function for an increasing function is increasing so g^{-1} is increasing. On the other hand, the composition of an increasing and a decreasing function is a decreasing function therefore $g^{-1}(f)$ is a decreasing function, which proves that NI is decreasing. From assumption $g(-1) + g(1) = 2g(0)$ we get $N(0) = g^{-1}(2g(0) - g(0)) = g^{-1}(g(0)) = 0$. Now, we prove that $x \cdot N(x) \le 0$. Because for $x = 0$ it obvious case. Let us consider two cases. First, if $x > 0$, so then from monotonicity, we have $g(x) > g(0)$. Then, $g(1) + g(-1) - g(x) = 2g(0) - g(x) \le g(0)$. Since, g^{-1} is increasing then

$g^{-1}\big(g(1) + g(-1) - g(x)\big) \le g^{-1}(g(0)) = 0$. Therefore, $x \cdot g^{-1}(g(1) + g(-1) - g(x)) \le 0$. Similarly, for $x < 0$, then from monotonicity, we get $g(x) < g(0)$, $g(1) + g(-1) - g(x) = 2g(0) - g(x) \ge g(0)$. Since, g^{-1} is increasing then $g^{-1}\big(g(1) + g(-1) - g(x)\big) \ge g^{-1}(g(0)) = 0$. So, $x \cdot g^{-1}(g(1) + g(-1) - g(x)) \le 0$.

Example. Let be given the function

$$g(x) = \begin{cases} x & x \in [0,1] \\ \sqrt[3]{x} & x \in [-1,0) \end{cases}, \quad g^{-1}(x) = \begin{cases} x^3 & x \in [-1,0) \\ x & x \in [0,1] \end{cases}$$

fulfils the assumption of Theorem 6. On the basis on Eq. (3) we get

$$NI(x) = \begin{cases} -x^3 & x \in [0,1] \\ -\sqrt[3]{x} & x \in [-1,0) \end{cases}.$$

Hence, such theorems are important in modelling decisions under uncertainty, where signal reversal has prognostic significance. This theorem has implications for decision modelling under uncertainty, where signal reversals have predictive value. From another perspective, this approach recognises that machine learning systems that adapt to changing input conditions need.

Theorem 7. If there exists a strictly increasing and continuous function $g: [-1,1] \to \mathbb{R}$ such that $g(0) = 0$, then

$$NI(x) = g^{-1}\big(-g(x)\big), \qquad for\ all\ x \in [-1,1].$$

Proof. From assumption we get $NI(0) = g^{-1}(-g(0)) = 0$. Since g is a strictly increasing function, the function $-g$ is strictly decreasing. The inverse function g^{-1} is strictly increasing. Therefore, the composition of strictly decreasing and increasing functions give us a strictly decreasing function. So, $NI(x) = g^{-1}(-g(x))$ is a strictly decreasing function.

Now, let us check the condition $x \cdot NI(x) \le 0$. Firstly, if $x > 0$, then monotonicity we get

$$0 < x \Rightarrow (0 = g^{-1}(-g(0)) > g^{-1}(-g(x))).$$

Secondly, if $x < 0$, then monotonicity we get $x < 0 \Rightarrow (g^{-1}(-g(x)) > g^{-1}(-g(0)) = 0)$. Consequently, $NI(x) < 0$.

COMPLEMENT OF A FUZZY BALANCED SET

In crisp sets, the complement of a set A is formed by excluding from the space X elements that belong to A. The distance between the value of belonging to and not belonging to the set A is always 1. Thus, the term says that the set A (thus also its complement A') is determined by the elements that belong to it. Firstly, let us emphasise the motivation for determining the supplementary set, which shows how much the elements under study lack to belong or not belong to a given set fully, and we introduce a definition:

Definition 2. [[6], Definition IV.1] The supplement of the fuzzy balanced set A is the set denoted by A^S satisfying the condition:

$$\eta_{A^S}(x) = I(\eta_A(x)).$$

However, we determine the complement of the set by indicating those elements which belong to the opposite class. On the other hand, the set of those elements that do not belong to either class is left in the neutral set.

Definition 3. [[6], Definition IV.6] The complement of the fuzzy balanced set A is the set denoted by A^C satisfying the condition

$$\eta_{A^C}(x) = NI(\eta_A(x))$$

for any $x \in X$.

To sum up, in BFS, complement is induced by a balanced negation NI, while supplement is induced by a fuzzy negation n. Let us recall that a crisp set and a fuzzy set are empty when, for each element $x \in X$, the membership function is identically equal to 0 and otherwise not empty. Thus, a set is empty if no element belongs to it. In balanced fuzzy sets, if an element does not belong to a given set, it may belong to its complement or be a neutral element. Therefore, a set whose all values of η belong to the interval $[-1, 0]$ is called empty and denoted by $\emptyset$. A set is non-empty when at least one element x, $\eta(x) \in (0, 1]$. If for every $\eta(x) = 0$, the set is called full neutral. A set is deeply empty if and only if its membership function is identically -1. It is denoted by $\emptyset_{\emptyset}$. They contribute to the creation of a more coherent theoretical and practical framework. It is worth noting that new fuzzy trust models can be built on this basis, although they are no longer rule-based; they typically aggregate [0,1] scores without explicitly separating trust from distrust. This could improve trust systems such as those described in [14].

Example. Let the membership function η describe the degree of profitability of a financial operation. It takes on positive values for profitable operations, a value of zero for economically neutral operations and negative values for loss-making operations. If we assume that $\eta(x) \leq 0$ denotes a financially non-profitable operation, then a value of -1 indicates the most unfavourable process. If the set of these operations is deeply empty, it means that not all operations are in the class of operations that can bankrupt the firm. If the set is only empty, it means that there are operations that are more or less disadvantageous. There may be financially neutral operations for the firm.

Example. Let us consider the following system.

Inputs (already mapped to [-1, 1]):

d_1: air dehumidification level read from the device,
d_2: humidity level of the outside air supply.

Output:

d_3: control level (in [-1, 1])
d_3>0: increase intensity,
d_3<0: decrease intensity,
d_3 = 0: no change.

In the rule:

$$IF d_1 OR\ NOT d_2\ THEN\ d_3,$$

where NOT is used as reversal operator N (see Tab. 3) and as OR is used balanced fuzzy t-conorm MAX [7]. Therefore, we get three activation indicators: Increase, Decrease, NoChange with an example activation signal: $sig = MAX(d_1, N(d_2))$. For example, assume the following input values: $d_1 = -0.4, d_2 = -0.7$. Then $N(d_2) = -(-0.7) = 0.7$. The balanced MAX function returns the argument with the larger absolute value: $sig = MAX(d_1, N(d_2)) = MAX(-0.4, 0.7)$. Because the positive signal 0.7 dominates over the negative signal, the activation corresponds to the rule indicating an increase. Therefore, the increase rule is activated.

CONCLUSIONS

This study explored the concept of balanced fuzzy negation as an essential operator in the framework of balanced fuzzy sets. The proposed approach extended classical fuzzy negation by introducing symmetry about the origin and distinguishing between membership, non-membership, and neutrality.

Extending fuzzy negations to balanced fuzzy negations ensures that neutral values (0) remain stable reference points. For example, in a patient health monitoring system, a balanced assessment allows for a clear distinction between a patient with normal vitality (0) and those with improvement or deterioration, while maintaining interpretive neutrality.

Furthermore, in opinion mining, when opinions are re-evaluated using multiple models, the final result remains directionally consistent – negativity does not randomly change to positivity through repeated processing.

Balanced fuzzy negation is defined by its fundamental properties, such as monotonicity, continuity, and involution. Fundamental construction theorems are presented, demonstrating that balanced negations can be systematically generated from classical fuzzy negations. Examples based on power, trigonometric, and Sugeno functions illustrate the flexibility of this approach.

A clear distinction is made between the complement and the complement of a balanced fuzzy set. While complementation measures the degree to which an element is not a full member, complementation identifies elements belonging to the opposite class. This distinction provides a more sophisticated way to represent and interpret uncertainty and bipolar information in the interval $[-1,1]$.

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