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Discrete interpolation based on the area of possible location of the evolute of a monotone curve

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ABSTRACT

Methods for geometric modelling of curves with a given set of properties interpolating point series of complex configuration form the foundation for developing computer-aided design systems for products bounded by functional surfaces. The key characteristics of the interpolating curve, which ensure the necessary surface properties, include a regular change in curvature values and a minimum number of singular points. The article aims to develop a method for generating a sequence consisting of an arbitrarily large number of specified reference points and assigned intermediate points, which can be interpolated by a monotone curve. The positions of intermediate points are determined based on the pre-assigned properties of the interpolating curve, including the positions of normals and curvature values. The correctness of the solutions proposed in the article is validated through the resolution of a test example. The method developed in the paper is a crucial step towards solving the problem of forming a contour that represents, with given accuracy, a curve with specified properties, interpolating a point series of arbitrary configuration.

Keywords: interpolation, monotone curve, intermediate points, normal, curvature centre, evolute, curvature radius.

INTRODUCTION

Computer-aided design (CAD) systems are a crucial component of modern manufacturing. Their use contributes to improving product quality while reducing costs and time. With the increase in computing capacity, the fundamental task for the effective use of CAD is the development of more advanced mathematical support for CAD/CAM/CAE systems. Methods for geometric modelling of curves with a given combination of properties interpolating point series of complex configuration form the foundation for developing CAD systems for products bounded by functional surfaces [1, 2]. Geometric modelling of surfaces determines the functional qualities of numerous objects. They include, in particular, products with aerodynamic and hydrodynamic contours in industries like aerospace, automotive, shipbuilding, power, and chemical engineering [3–5]. Complex surfaces are often modelled on the basis of linear frameworks, where the elements are formed by interpolating series of points located on the surface. At the same time, the geometric properties of the interpolating curve ensure product performance. For the products mentioned, the primary functional characteristic is the specified flow of the surface with the environment. The key characteristics of the interpolating curve which ensure the necessary surface properties are a regular change in curvature values and a minimal number of singular points [6, 7].

Interpolation methods can be divided into two groups depending on how the information is represented. The first group includes methods of continuous geometric modelling, where the result is a model represented by a function or a set of functions. In these methods, the problem of forming an interpolating curve is addressed by creating a contour—a curved line consisting of sections of analytically defined curves connected at the reference points.

When forming an interpolating curve from arcs of second-order curves [8, 9], the uncontrolled occurrence of inflection points is guaranteed to be avoided; however, it is impossible to eliminate points with extreme curvature values completely. The shape of the curve can be adjusted by changing the positions of the tangents at the reference points. If the positions of the tangents at the reference points are fixed, local adjustments can be made using arcs of ellipses.

Using B-splines [10, 11] provides greater control over the shape of the contour. A B-spline is defined by a set of reference points, each corresponding to a conjugate function. The curve approximates a polygonal line connecting the reference points. The smoothness of the generated contour is determined by the degree of the conjugate functions. As smoothness increases, the degree of the conjugate functions and the number of vertices in the defining polygon also increases. This reduces the ability to adjust the curve shape locally. At the same time, the likelihood of oscillations increases [12, 13].

Improving the junctions between contour segments requires increasing the spline degree, leading to more reference points. This complicates the process of local adjustment of the spline shape and parameters. Additionally, when interpolating a large number of points, ensuring a monotonic change in curvature along the B-spline becomes challenging.

The absence of oscillation in a B-spline-based contour can be ensured by controlling the shape of the defining polygon. Modern CAD system packages allow controlling the shape of the defining polygon of the B-spline in interactive mode [14, 15]. When the number of reference points is large, it becomes difficult, if not impossible, to ensure the absence of oscillation. Furthermore, when forming a curve using a B-spline, there is no mechanism to control the dynamics of curvature changes along the contour. These peculiarities limit the ability to ensure the specified characteristics of the curves formed using B-splines.

The advantage of the continuous geometric modelling methods discussed is the analytical description of the curve sections, which enables the unambiguous definition of the formed contour and its characteristics at the nodes. However, this also imposes the properties of the curves with sections modelled as a contour on the curve interpolating the point series.

An increase in the number of conditions imposed on the geometric image formed using such methods requires an increase in the parametric number of curves that make up its determinant. This leads to the uncontrolled occurrence of singular points on the curves, thus reducing the quality of the obtained solution. At present, no definitive approach to addressing this issue exists.

The second group includes discrete geometric modelling, which provides a solution in the form of a structured set of points [16, 17]. This approach allows for controlling the characteristics of the interpolating curve by abandoning its analytical representation and using algorithms that determine the positions of intermediate points of the curve as a shaping tool. However, to this day, this potential capability of discrete geometric modelling has largely remained unimplemented. The main challenges that need to be addressed to improve the efficiency of the discrete approach are controlling the geometric properties of the curve through the characteristics of the point series that belong to it and estimating the accuracy of discrete interpolation. The purpose of the discrete interpolation method developed by the authors of this article is to form a series consisting of the original and newly assigned intermediate points, which can be interpolated by a curve with specified characteristics. The primary characteristic of the curve interpolating the point series generated by our method is the minimum number of singular points determined based on the input data. The following discrete interpolation scheme is proposed:

1. The original point series is divided into segments that can be interpolated by a curve containing no singular points. Such a curve is referred to as monotone because the curvature values change regularly and monotonically along the curve. Any curve can be regarded as a combination of monotone segments joined at singular points;
2. Sequences consisting of any number of points are formed, defining the monotone segments of the interpolating curve. These are the reference points and intermediate points assigned during the modelling process. The process of forming the specified point series will be called densification. The newly assigned intermediate points will be termed densification points. Densification is carried out until the discrete interpolation error becomes less than the specified value. This error will be estimated by the size of the area of possible locations of monotone interpolating curve segments;
3. A contour composed of continuous line segments is formed, which interpolates the densification point series and is located within the area of possible location of the monotone curve. Such a contour represents a curve with specified geometric characteristics with required accuracy.
4. The problem of forming the area of possible location of a monotone interpolating curve as a sequence of closed contours joined at the reference points was solved in [18, 19]. The area of possible location of the curve segment *i…i+1* is bounded by arcs of circles. These are the circles osculating to the curve at the reference points (OC_i and OC_{i+1}) and the circles that touch the osculating circle and the tangent line at another reference point (Figure 1).

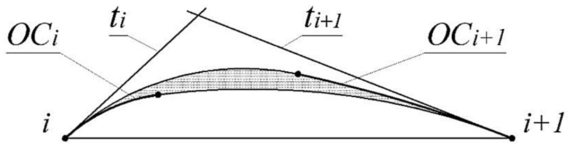

Figure 1. Area of possible location of a segment of a monotone interpolating curve

The boundaries of the area of location of the section are unambiguously defined by the position of the curvature centres of the monotone curve, which correspond to the points that bound the section. The problem of determining the positions of the curvature centres corresponding to the reference points for the monotone curve interpolating the range … *i-1*, *i*, *i+1*… is solved in the article [20]. Each curvature centre is assigned within an area, the boundaries of which are defined based on the analysis of the mutual arrangement of the adjacent, tangent and osculating circles defined by the points belonging to the monotone curve.The osculating circle OC_i touches the curve at point *i*, and its radius is the inverse of the curvature value of the curve at that point [21].

The tangent circle is defined by passing through point *i*, touching line t_i (tangent to the curve at point *i*) and passing through some point that belongs to the curve (Figure 2).

The circle which is tangent to the curve at point *i* and passes through a point located outside OC_i is labelled TC'_i. The tangent circle passing through a point located inside OC_i, is labelled $'TC_i$. In the figure, the centres of OC_i and TC'_i are marked as C_i and O'_i, respectively.

By analysing the location and dimensions of the adjacent and tangent circles, which are defined by the point series assigned on the curve along which the values of the curvature radii increase monotonically, we obtained the following relation:

$$\ldots < R_{i-1} < RTC'_{i-1} < R\,TC_i < R_i < \\ < RTC'_i < R'\,TC_{+1} < R_{i+1} < \ldots, \quad (1)$$

where: R_i, $R\ 'TC_i$, $R\ TC'_i$ are the radii of OC_i, $'TC_i$, TC'_i, respectively.

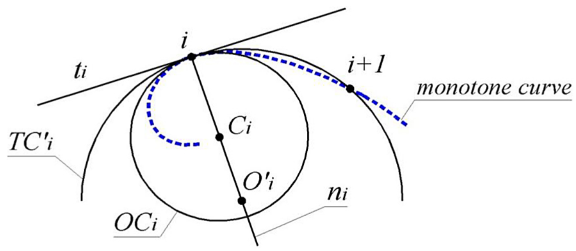

Figure 2. Location of the osculating and tangent circles relative to the monotone curve

The adjacent circle (AC_i) passes through three consecutive points *i-1, i, i+1,* which are assigned on the curve (Figure 3). By analysing the location and dimensions of the adjacent and tangent circles, the following relation is derived:

$$\ldots < R\,AC_{i-1} < R\,TC'_{i-1} < R'\,TC_i < RAC_i < \\ < RTC'_i < R'\,TC_{i+1} < R\,AC_{i+1} < \ldots, \quad (2)$$

where: $R\,AC_i$ is the radius of AC_i.

The condition under which a monotone curve can interpolate the point series is derived from relation (2): $\ldots < R\,AC_{i-1} < R\,AC_i < \, < R\,AC_{i+1} < \ldots$

Relations 1 and 2 define the ranges within which the normals and curvature centres of the monotone interpolating curve, corresponding to the reference points, must be located.

The positions of the centres of AC_{i-1}, AC_i, AC_{i+1} (points S_{i-1}, S_i, S_{i+1}, respectively) determine the boundaries of the possible location of the normal n_i of the curve at the reference point *i*. To fulfil (2), normal n_i must intersect segments [S_{i-1}, S_i] and [S_i, S_{i+1}] simultaneously. Points O_i and $O'_{I,}$ where the normal intersects these segments, are the centres of $'TC_i$ and TC'_i, respectively. At the same time, to fulfil (1), normal n_{i-1} must intersect segment [S_i, S_{i+1}], and normal n_{i+1} must intersect segment [O'_i, S_{i+1}]. To achieve a uniform increase in curvature values along the formed curve, the optimal position of normals n_i and n_{i+1} is where they divide segment [S_i, S_{i+1}] into equal parts: $|S_i, O'_i| = |O'_i, 'O_{i+1}| = |'O_{i+1}, S_{i+1}|$. Paper [20] suggests an algorithm for the simultaneous assignment of the positions of normals at all reference points, at which they divide the corresponding segments bounded by the centres of adjacent circles in a proportion close to the optimum.

The assigned position of the normal automatically determines the area of possible locations of the curvature centre belonging to it. According to (1), for normal n_i, this range corresponds to segment [$'O_i$, O'_i]. The positions of the curvature centres are assigned within the specified ranges taking into account the properties of the evolute of the monotone curve [21]:

- the evolute is a convex curve;
- the normal of the curve are the tangents of its evolute at the corresponding curvature centres;
- the length of any segment of the evolute is equal to the difference of the values of the curvature radii at the points bounding the corresponding segment of the original curve.

The normals of the interpolating curve, assigned at the reference points, and the curvature centres belonging to them define a sequence of basis triangles (BT_i) (Figure 4).

The method of assigning the positions of the curvature centres of the interpolating curve corresponding to the reference points ensures that the relation for each of the triangles is fulfilled:

$$|C_i, C_{i+1}| \le R_{i+1} - R_i \le |C_i, T_i| + |C_{i+1}, T_i| \quad (3)$$

Relation 3 is a necessary condition for forming a segment of the evolute of the monotone curve that interpolates the given reference points within each of the basis triangles. The resulting area of location of the evolute defines the interpolating curve with accuracy determined by the original point series. In practical applications, however, this level of accuracy may prove insufficient. The aim of this article is to develop and test a method for forming a new sequence consisting of an arbitrarily large number of reference and intermediate points, which represent a monotone curve, based on the original area of location of the evolute of the interpolating curve with a given accuracy. To achieve this aim, the following steps are required:

- Developing a method for assigning the positions of normals and curvature centres corresponding to the intermediate points of the curve;
- Developing a method for determining the positions of intermediate points corresponding to

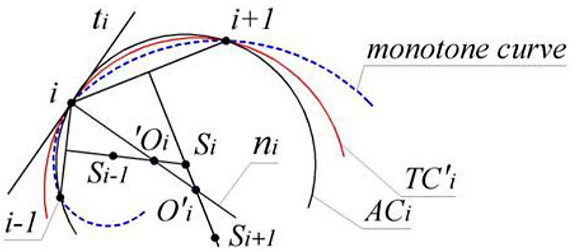

Figure 3. Location of adjacent and tangent circles

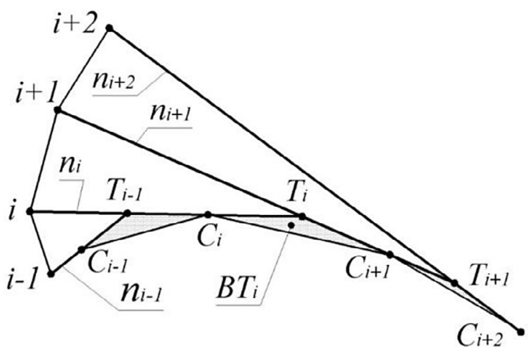

Figure 4. Area of location of the evolute of the discretely presented curve

the assigned characteristics of the interpolating curve;
- Suggesting a method for determining the absolute error with which the generated sequence of points represents a monotone curve;
- Testing the suggested methods while determining the positions of intermediate points for the original point series assigned on the monotone curve.
- To develop the above methods, the following tasks need to be addressed:
- Determining the area of possible location for the normal of the interpolating monotone curve at an intermediate point;
- Identifying the area of possible location for the curvature centre corresponding to the intermediate point;
- Defining the area of location for the intermediate point.

MATERIALS AND METHODS

After assigning the curvature centres for the reference points, densification can be performed locally on separate sections and in any order. The condition for assigning a densification point is the presence of an area within which a monotone curve interpolating the formed point series can exist. Each densification step involves sequential determination of the positions of the normal, the curvature centre, and the corresponding densification point. Each of these elements is assigned within the area, the boundaries of which are defined by the characteristics of the interpolating curve. Having assigned the normal n_{cn} corresponding to the densification point of the i-th segment (i_{cn}) and the curvature centre (C_{cn}) located on this normal, within the original BT_i, we obtain two new basis triangles – C_i,C_{cn},T_1 ($'BT_{cn}$) and C_{cn},C_{i+1},T_2 (BT'_{cn}). From the condition that the location of the evolute segment of the monotone

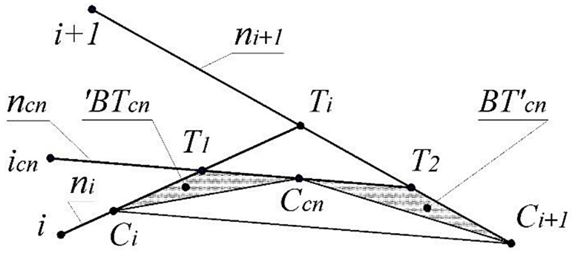

Figure 5. Densification of a segment of the reference point series

curve within $'BT_{cn}$ and BT'_{cn}, the following relationship is derived (Figure 5):

$$|C_i,C_{cn}| + |C_{cn},C_{i+1}| \leq R_{i+1}-R_i \leq \\ \leq |C_i,T_1| + |T_1,T_2| + |T_2,C_{i+1}| \quad (4)$$

Let us determine the boundaries of the area within which normal n_{cn} must be located in accordance with condition (4). Simultaneously, we will impose the requirement that n_{cn} remains parallel to line (C_i,C_{i+1}). This restriction is correct because the evolute segment (C_i,C_{i+1}) is a convex curve. This guarantees the existence of a tangent line at some point of the evolute segment that is parallel to the chord [C_i,C_{i+1}].

The position of n_{cn}, where the distance between the normal and the line (C_i,C_{i+1}) is minimised, is determined by the following relations:

$$\begin{cases} |C_i,T_1|+|T_1,T_2|+|T_2,C_{i+1}|=R_{i+1}-R_i \\ |C_i,T_1|:|C_i,T_i|=|C_{i+1},T_2|:|C_{i+1},T_i| \end{cases} \quad (5)$$

Based on (5) and the similarity of triangles C_i,C_{i+1},T_i and T_1,T_2,T_i, the minimum distance between n_{cn} and the line (C_i,C_{i+1}) (h_{min}) can be expressed through the parameters of the original BT_i:

$$h_{min} = \frac{2S(R_{i+1}-R_i-c)}{c(a+b-c)} \quad (6)$$

where: S is the area of location of BT_i, $a = |C_i,T_i|$, $b = |C_{i+1},T_i|$, $c = |C_i,C_{i+1}|$.

In the case when the distance between n_{cn} and (C_i,C_{i+1}) is h_{min}, the polygonal line $C_i – T_1 – T_2 – C_{i+1}$ represents the unique configuration of the evolute for the segment of the monotone curve (i, $i+1$), and this segment is composed of two smoothly connected arcs of circles with centres at points T_1 and T_2. The maximum possible distance (h_{max}) between n_{cn} and (C_i,C_{i+1}) is equal to the height of the triangle C_i,C_{i+1},C_{cn}, the sides of the triangle are defined by the following relations:

$$\begin{cases} |C_i,C_{cn}|+|C_{cn},C_{i+1}|=R_{i+1}-R_i, \\ |C_i,C_{cn}|=|C_{cn},C_{i+1}|. \end{cases} \quad (7)$$

The distance between point C_{cn} and line (C_i,C_{i+1}) cannot exceed the value of h_{max}, which can be calculated by the formula:

$$h_{max} = \frac{1}{2}\sqrt{(R_{i+1}-R_i)^2-|C_i,C_{i+1}|^2} \quad (8)$$

If the distance between point C_{cn} and line (C_i,C_{i+1}) is h_{max}, then the monotone curve segment (i, $i+1$) consists of three smoothly connected arcs of circles with centres at points C_i, C_{cn}, C_{i+1}.

Assigning a n_{cn} with a distance from (C_i,C_{i+1}) within the range $h_{min}<h_{cz}<h_{max}$ allows us to form a section of the evolute as a smooth convex line. Such an evolute defines the involute with a regular and monotonic change in curvature.

After assigning the position of the normal n_{cn}, the area of possible location for the curvature centre C_{cn} is determined. This area is the segment $[C_1, C_2]$, where C_1 is the position of point C_{cn} at which the curvature radius (R_{cn}) at the densification point is minimum (of the possible positions of C_{cn} point, C_1 is closest to point i_{cn}); C_2 is the position of the curvature centre at which R_{cn} is maximum. The position of points C_1 and C_2 is determined based on the condition that $|C_i,C_{cn}|+|C_{cn},C_{i+1}|=R_{i+1}-R_i$. The fulfilment of this condition means that in the case when R_{cn} is maximum or minimum, the evolute of the curve segment is a polygonal line $C_i–C_{cn}–C_{i+1}$, and the geometric location of the positions of point C_{cn} follows an ellipse with focal points at points C_i, C_{i+1}, with the length of the major axis equal to $R_{i+1}-R_i$ [22]. In a rectangular Cartesian coordinate system with the origin at the centre of the segment $[C_i, C_{i+1}]$ and the positive direction of the abscissa axis coinciding with the vector $\overrightarrow{C_i,N_{i+1}}$, this ellipse is defined by the following equation:

$$\frac{x^2}{(R_{i+1}-R_i)^2}+\frac{y^2}{(R_{i+1}-R_i)^2-|C_{i+1},C_i|^2}=\frac{1}{4} \quad (9)$$

For different positions of the normal n_{cn}, the boundaries of the range $[C_1, C_2]$ are obtained at the intersection of the normal and the ellipse (Figure 6). If points C_1 or C_2 lie outside BT_i, then the boundary of the area of location of C_{cn} is point T_1 or T_2, respectively.

The problem of generating a point series belonging to a monotone curve has a solution when the curvature centre is assigned at any point of the segment $[C_1, C_2]$. If this assignment does not contradict any additional conditions imposed on the generated curve, C_{cn} is assigned at the centre

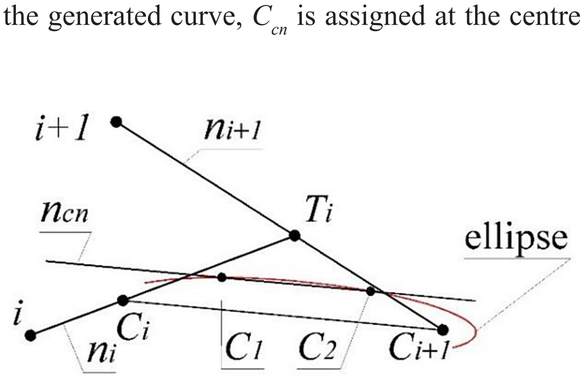

Figure 6. Determining the area of location of the curvature centre

of $[C_1,C_2]$. The area of location for the densification point i_{cn} is defined on the normal n_{cn} based on the assigned position of the curvature centre C_{cn}. This area is bounded by the maximum (R_{cn}^{max}) and minimum (R_{cn}^{min}) values that the curvature radius of the interpolating curve can take at the densification point:

$$R_{cn}^{min} \le R_{cn} \le R_{cn}^{max} \quad (10)$$

The location of the evolute of the monotone curve within $'BT_{cn}$ and BT'_{cn} means that:

$$\begin{cases}|C_i,C_{cn}|\le R_{cn}-R_i \le |C_i,T_1|+|T_1,C_{cn}|;\\ |C_{i+1},C_{cn}|\le R_{i+1}-R_{cn}\le |C_{cn},T_2|+|T_2,C_{i+1}|.\end{cases} \quad (11)$$

Based on (11), the lower boundaries of area (10) R_{cn}^{min} must be the larger of the values of $R_i+|C_i, C_{cn}|$ or $R_{i+1}-(|C_{i+1},T_2|+|T_2,C_{cz}|)$. The upper boundary of the area R_{cn}^{max} is equal to the smaller of the values of $R_i + |C_i,T_1| + |T_1,C_{cn}|$ or $R_{i+1} - |C_{i+1},C_{cn}|$. Let us demonstrate that the proposed scheme for determining the position of the normal n_{cn} and defining the boundaries of the area of location of the curvature centre $[C_1, C_2]$ guarantees the existence of the area (10) for any ratio of the values selected for R_{cn}^{min} and R_{cn}^{max}.

If the curvature centre takes the extreme position $C_{cn}\equiv C_2$, then the values of $R_i+|C_i,C_{cn}|$ and $R_{i+1}-|C_{i+1},C_{cn}|$ are equal. Clearly, in this case, $|C_{i+1},C_{cn}|<|C_{i+1},T_2|+|T_2,C_{cn}|$ (Figure 5), and consequently,

$$R_i+|C_i,C_{cn}| > R_{i+1}-(|C_{i+1},T_2|+|T_2,C_{cn}|) \quad (12)$$

When C_{cn} is shifted from C_2 towards C_1, the distance of $|T_2,C_{cn}|$ increases at a faster rate than $|C_i,C_{cn}|$ decreases. This ensures that relation (12) remains valid, and the value of $R_{cn}^{min}=R_i+|C_i,C_{cn}|$. A similar line of reasoning proves the following:

$$R_{cn}^{max}=R_{i+1}-|C_{i+1},C_{cn}| \quad (13)$$

Based on the relations $|C_i,C_{cn}| < |C_i,C_2|$, $|C_{i+1},C_{cn}|>|C_{i+1},C_2|$ and $R_i+|C_i,C_2|=R_{i+1}-|C_{i+1},C_2|$, it can be concluded that the location of the normal n_{cn} within the range $h_{min}<h_{cn}<h_{max}$, and of the curvature centre C_{cn} within the segment $[C_1, C_2]$ guarantees the fulfilment of the condition that $R_{cn}^{min}\le R_{cn}^{max}$, and thus ensures the existence of the area of possible locations of the densification point. The value of the curvature radius at the densification point is set equal to $R_{cn}=0.5\left(R_{cn}^{min}+R_{cn}^{max}\right)$. The position of the densification point is determined by drawing a segment from C_{cn}, whose length is equal to R_{cn}. Interpolating a point series consisting of an

arbitrarily large number of points by a monotone curve is possible if the characteristics of the basis triangles defining its evolute satisfy the requirement (3). The proposed scheme of assigning the normal n_{cn} and the curvature centre C_{cn} corresponding to the densification point ensures the formation of two new densification basis triangles inside the original basis triangle, the characteristics of which meet the requirement (3). Each basis triangle, either original or one obtained as a result of densification, defines the area of possible location of the corresponding segment of the interpolating curve [18]. Preserving the specified characteristics of the basis triangles formed during any number of densifications allows us to assert that the consistent assignment of normals, curvature centres and densification points within the ranges that take into account the area of possible solution of the problem guarantees the formation of a point series that can be interpolated by a monotone curve.

RESULTS AND DISCUSSIONS

Let us consider the solution to the problem of forming a point series that can be interpolated by a monotone curve, using the example of densification of a point series assigned on a branch of a parabola defined by the equation $y = \frac{x^2}{300}$. As a reference, we take the point series used in [20] to determine the area of location of the evolute of the monotone curve. The characteristics of the reference point series are provided in Table 1.

After assigning the normals and curvature centres of the monotone curve corresponding to the reference points, the area of location of the evolute of the monotone curve is obtained. The characteristics of the original area of location of the evolute are given in Table 2. The order of densification of the point series will be demonstrated using the example of the third reference segment.

The location range of the normal of the monotone curve at the densification point is determined by the minimum distance h_{min}= 0.887 mm and the maximum distance h_{max}= 1.258 mm between the normal and the base of BT_3. After assigning the position of the normal at the centre of the specified range, the range of the curvature centre of the monotone curve at the point of densification is determined as $|C_1, C_2|$ = 9.873 mm. The assigned positions of the normal and the curvature centre corresponding to

Table 1. Characteristics of the reference point series

Point coordinates, mm		Length of the chord of the supporting polygonal line, mm	The radius of the osculating circle, mm	Sector of location of the normal, °
x	y	h_i	$R\ AC_i$	Δn_i
30	3	31.32	-	-
60	12	33.54	188.58	1.074
90	27	36.62	238.73	1.074
120	48	84.85	348.37	1.386
180	108	103.23	570.09	2.727
240	192	123.55	1002.64	2.244
300	300	-	-	-

Table 2. Characteristics of the reference point series

Radius of the tangent circle, mm	Curvature radius of the monotone curve, mm	Location range of the normal, mm	Location range of the curvature centre, mm	Location range of the densification point, mm
$R'TC_i$	$R\ TC'_i$	R_i	h_{max} - h_{min}	$R^{max}_{\tilde{n}n} - R^{min}_{\tilde{n}n}$
-	166.74	155.10	0.274	6.782
176.10	201.94	184.56	0.312	7.895
218.66	260.65	239.79	0.371	9.873
319.22	384.88	350.61	0.428	38.581
468.62	693.55	569.53	0.527	52.231
837.83	1200.0	986.86	0.863	112.846
1422.14	-	1729.04	-	-

Table 3. Characteristics of the densification point series

i	Point coordinates, mm		Length of the chord of the supporting polygonal line, mm	Radius of the adjacent circle, mm	Radius of the tangent circle, mm	Curvature radius of the monotone curve, mm
	x	y	h_i	$R\ AC_i$	$R'TC_i$	R_i
1	30	3	14.62	-	-	153.27
2	44.20	6.50	16.73	167.13	161.14	173.61
3	60.00	12.00	15.44	187.53	182.48	196.50
4	74.10	18.30	18.12	212.79	205.80	221.59
5	90.00	27.00	17.28	237.99	231.11	247.40
6	104.50	36.40	19.36	266.25	258.71	277.96
7	120.00	48.00	18.03	352.14	342.08	358.21
8	148.30	73.40	46.93	428.94	392.05	474.50
9	180.00	108.00	45.19	564.34	519.17	626.10
10	207.70	143.70	58.10	741.47	704.36	800.27
11	30.00	3.00	14.62	-	-	153.27
12	44.20	6.50	16.73	167.13	161.14	173.61
13	60.00	12.00	15.44	187.53	182.48	196.50

the densification point determine the area of its possible location, the value of which is equal to $R_{\tilde{n}n}^{max} - R_{\tilde{n}n}^{min} = 0.009$ mm.

After assigning the densification points of the other segments using a similar scheme, a point series was obtained, the characteristics of which are given in Table 3. As a result of densification, a new sequence of points has been generated. The values of the radii of the adjacent circles corresponding to these points increase steadily, satisfying the necessary condition for their subsequent interpolation by a monotone curve. The normals assigned at the reference points and densification points define a new sequence of tangent circles. The ratios of the radii values of the adjacent, tangent and osculating circles adhere to conditions (1) and (2), which validates the correctness of the assigned characteristics of the interpolating curve.

The assignment of intermediate points results in localising the area of possible location of the curve interpolating the new sequence of points. Moreover, this new localised area of possible location of the interpolating curve and the original curve are both located within the original area.

To estimate the absolute error with which the point series resulting from the densification defines the interpolating curve, the location range of the densification point (10) can be utilised. Once the range length for each of the segments of the densified point series falls below the specified value, the sequence is considered complete. Depending on the task requirements, the interpolating curve can be represented either by the area of possible location or as a contour. In the latter case, within the area of possible location of the interpolating curve, a smooth contour composed of arcs of circles [18], ellipses or a B-spline [19] is formed. The methodology proposed in [20] provides the highest accuracy in representing the interpolating curve. To model the curve using the developed method, any point series where the radii of adjacent circles increase or decrease monotonically can be used as input data.

To demonstrate how the developed method works, the simplest way to generate a sequence of original points is to assign points along a monotone section of any plane curve. This could be a second-order curve, an involute of a circle, or a cycloid. Alternatively, the original point series can be constructed without relying on a specific curve by defining a sequence of intersecting arcs of circles whose radii satisfy the given conditions of the problem.

The parabola was selected as the initial curve due to its widespread use in various calculations and constructions, as well as its well-known and extensively studied properties. The positions of the reference points assigned on the parabola demonstrate that there are no restrictions on the distance ratios between the points or on their alignment to a grid. As a result of the conducted research, the algorithm has been proposed for forming a smooth contour that represents a curve interpolating a point series of arbitrary configurations with a specified accuracy.

1. The reference point series is divided into segments where the radii of osculating circles either increase or decrease. Each such segment defines a monotone curve.
2. For each reference point, the location range of the normal of the monotone curve is determined, and the position of the normal is assigned within this range. The positions of the curvature centres are assigned within intervals belonging to the corresponding normals, the boundaries of which ensure that condition (3) is satisfied. The assigned normals and the curvature centres define a sequence of basis triangles, which serve as the reference area for the possible location of the evolute of the interpolating curve.
3. Local densification of the original point series is carried out. The densification point is assigned within the range, which is determined by the position of the normal and the curvature centre previously assigned to the densification point, as well as the normals and the curvature centres that define the corresponding original basis triangle. The point series is considered fully formed once the location areas of the densification points across all segments fall below the specified interpolation accuracy.
4. The resulting sequence of basis triangles defines the area of possible location of the interpolating curve within which the contour is formed.

The developed method is based on analysing the configuration of the original point series and employs an algorithm for assigning intermediate points within the area of possible location of the curve with a monotonic change in curvature.

Through this analysis, the original point series is segmented into sections, each of which can be interpolated by monotone curves. These monotone segments are constructed independently, and the method imposes no constraints on their quantity. Regardless of modifications to the input data, all curve segments remain confined to a bounded spatial region. The specified characteristic of the method ensures its robustness against variations in the initial data. Increasing the number of original points and assigning intermediate points within the area of possible location of the interpolating curve leads to localisation of the region and its consistent convergence to a unique solution. These characteristics of the method ensure its stability and convergence.

CONCLUSIONS

The paper introduces a method for constructing a sequence consisting of specified reference points and assigned intermediate points, ensuring that the resulting sequence can be interpolated by a monotone curve. The initial data for determining the positions of the intermediate points include the coordinates of the reference points and the configuration of the area of possible location of the evolute of the interpolating curve. The area of the evolute is bounded by a sequence of basis triangles, which is defined by the characteristics of the interpolating curve – the normals assigned at the reference points and the curvature centres assigned on these normals.

The algorithm for assigning the position of an intermediate point is based on forming the area of the evolute of the monotone curve interpolating the obtained point series. The algorithm is based on the methods developed in this paper:

- assigning the positions of the normals of the interpolating curve corresponding to the intermediate points;
- assigning the positions of the curvature centres belonging to these normals;
- determining the positions of the intermediate points corresponding to the assigned normals and curvature centres.

For each intermediate point, the position of the normal is assigned within a range, the boundaries of which are unambiguously determined by the positions of the normals and the curvature centres assigned to the preceding and succeeding reference points. Assigning the normal within this defined range establishes the segment belonging to it, which is the area of possible location of the curvature centre of the monotone curve at the intermediate point.

As a result of assigning the normal and the curvature centre, two new basis triangles are formed within the original basis triangle, the configuration of which establishes the boundaries for the area of possible location of the intermediate point. Each of the triangles restricts the area of location of the evolute of the curve interpolating the area bounded by the assigned intermediate point and one of the two reference points (either preceding or succeeding).

Assigning a normal, a curvature centre or an intermediate point at the extreme of their respective allowable ranges results in the corresponding segment of the evolute being represented as a polygonal line, while the segment of the interpolating curve will consist of smoothly connected arcs of circles. By assigning the specified elements within the corresponding ranges, a new sequence of points is formed, along with the characteristics of a regular monotone curve that interpolates these points. Once the intermediate point has been assigned, it is considered as the reference point for assigning the subsequent points. The intermediate points can be assigned in any sequence on the original segments, and their final number can be arbitrarily large. All monotone curves interpolating the generated sequence of points and having assigned features at these points, pass within the area of possible location of an intermediate point on each of the segments. The length of these ranges can be used to estimate the absolute error with which the point series defines the interpolating curve. When the maximum absolute error falls below the specified value, the point series is considered to be formed.

The correctness of the proposed algorithm and its constituent methods have been examined by interpolating a point series assigned on the branches of a parabola. A sequence of 7 points was used as the reference, corresponding to the series utilised in [20] to establish the original area of location of the evolute of a monotone curve.

As a result of assigning an intermediate point, a new sequence of 13 points is formed within each of the segments bounded by the neighbouring reference points. The configuration of the obtained point series and the characteristics of the interpolating curve assigned at these points align with the conditions required for the points to belong to a monotone curve. This alignment ensures a further increase in the number of intermediate points while maintaining the ability to interpolate the formed point series by a monotone curve.

Assigning the positions of intermediate points and characteristics of the interpolating curve based on the area of possible location of its evolutes streamlines the geometric framework of the problem solution, minimizes the computational effort required, and ensures the desired level of accuracy in discrete interpolation.

The findings presented in this paper build upon and complement the results outlined in [18-20]. Consequently, a method has been developed for forming a contour that, with specified accuracy, represents a regular curve containing a minimum number of singular points and effectively interpolates a point series of arbitrary configuration.

The method provides a systematic approach to solving the following problems:

- The original point series is segmented into parts, each of which can be interpolated by a curve containing no singular points;
- Sequences consisting of reference and intermediate points are formed, which define the monotone segments of the interpolating curve with specified accuracy. The configuration of the generated sequence, containing both source and assigned intermediate points, ensures that they can be interpolated by a monotone curve and defines the area of possible location for the interpolating curve. Moreover, the distance between any admissible curves within this area cannot exceed the specified interpolation error tolerance;
- Areas of possible locations for the monotonic segments of the interpolating curve are formed;
- A contour consisting of continuous line segments is formed, which interpolates the obtained point series and lies within the area of possible location of the interpolating curve.

The method of interpolating a point series based on the area of possible location of the curve with monotonic change in curvature is universal. This approach guarantees interpolation with sections of monotone curves for any sequence of points located on the plane. The developed method proves most effective when applied to tasks requiring monotonic and regular curvature variation along the interpolating curve with interpolation accuracy within 10^{-3} mm tolerance (maximum CNC machining precision). These tasks include functional surface modelling, where the input data is obtained by determining the positions of the original points on a physical prototype.

REFERENCES

1. Okaniwa, S.; Nasri, A.; Lin, H.; Abbas, A.; Kineri, Y.; Maekawa, T. Uniform B-Spline Curve Interpolation with Prescribed Tangent and Curvature Vectors. IEEE Trans. Vis. Comput. Graph. 2016, 18, 1474–1487. https://doi.org/10.1109/TVCG.2011.262
2. Werghi, N.; Fisher, R.; Robertson, C.; Ashbrook, A. Object reconstruction by incorporating geometric constraints in reverse engineering. Comput. Aided Des. 1999, 31, 363–399. https://doi.org/10.1016/S0010-4485(99)00038-X

3. Cai, C.; Yang, Y.; Jia, Y.; Wu, G.; Zhang, H.; Yuan, F.; Qian, Q.; Li, Q. Aerodynamic load evaluation of leading edge and trailing edge windward states of large-scale wind turbine blade under parked condition. Appl. Energy 2023, 350, 121744. https://doi.org/10.1016/j.apenergy.2023.121744
4. Fooladi, M.; Foroud A.A. Recognition and assessment of different factors which affect flicker in wind turbine. IET Renew. Power Gener. 2015, 1, 250–259. http://doi.org/10.1049/iet-rpg.2014.0419
5. Hudym, V., Kosovska, V., Al_Issa, H., Shchur, T., Miroshnyk, O., Ziarkowski, S. Change of frequency characteristics of a filter using a reactor with smoothly adjustable inductance. Informatyka, Automatyka, Pomiary W Gospodarce I Ochronie Środowiska, 2024, 14(2), 28–33. https://doi.org/10.35784/iapgos.5810
6. Hashemian, A.; Imani, B.M. A new quality appearance evaluation technique for automotive bodies, including the effect of flexible parts tolerances. Mech. Based Des. Struct. Mach. 2018, 46, 157–167. https://doi.org/10.1080/15397734.2017.1321487
7. Mo, W.; Li, D.; Wang, X.; Zhong, C. Aeroelastic coupling analysis of the flexible blade of a wind turbine. Energy 2015, 89, 1001–1009. https://doi.org/10.1016/j.energy.2015.06.046
8. Badariyah, N.; Latip, A.; Omar, R. Feasible Path Generation Using Bezier Curves for Car-Like Vehicle. IOP Conf. Ser.: Mater. Sci. Eng. 2017, 226, 012133. https://doi.org/10.1088/1757-899X/226/1/012133
9. Hewett, D.P.; Ockendon, J.R.; Smyshlyaev, V.P. Contour integral solution of the parabolic wave equation. Wave Motion 2019, 84, 90–109. https://doi.org/10.1016/j.wavemoti.2018.09.015
10. Hashemian, A.; Hosseini, S.F. An integrated fitting and fairing approach for object reconstruction using smooth NURBS curves and surfaces. Comput. Math. with Appl. 2018, 76(7), 1555–1575. http://doi.org/10.1016/j.camwa.2018.07.007
11. Hoschek, J.; Müller, R. Turbine blade design by lofted B-spline surfaces. J. Comput. Appl. Math. 2000, 119(1–2), 235–248. https://doi.org/10.1016/S0377-0427(00)00381-2
12. Lezhenkin, O.M.; Halko, S.V.; Miroshnyk, O.O.; Vershkov, O.O.; Suprun, O.M.; Shchur, T.G.; Kruszelnicka, W.; Kasner, R. Investigation of the separation of combed heap of winter wheat. J. Phys. Conf. Ser. 2021, 1781(1), 012016. https://doi.org/10.1088/1742-6596/1781/1/012016
13. Tabor, S.; Lezhenkin, A.; Halko, S., Miroshnyk, A.; Kovalyshyn, S.; Vershkov, A.; Hryhorenko, O. Mathematical simulation of separating work tool technological process. E3S Web of Conferences. 2019, 132, 01025. https://doi.org/10.1051/e3sconf/201913201025
14. Li, W.; Xu, S.; Zheng, J.; Zhao, G. Target curvature driven fairing algorithm for planar cubic B-spline curves. Comput. Aided Geom. Des. 2004, 21, 499–513. https://doi.org/10.1016/j.cagd.2004.03.004
15. Massarwi, F.; Elber, G. A B-spline based framework for volumetric object modeling. Comput. Aided Des. 2016, 78, 36–47. https://doi.org/10.1016/j.cad.2016.05.003
16. Havrylenko, Y.; Cortez, J.I.; Kholodniak, Y.; Alieksieieva, H.; Garcia, G.T. Modelling of surfaces of engineering products on the basis of array of points. Teh. Vjesn. 2020, 27(6), 2034–2043. https://doi.org/10.17559/TV-20190720081227
17. Havrylenko, Y.; Kholodniak, Y.; Vershkov, O.; Naidysh A. Development of the method for the formation of one-dimensional contours by the assigned interpolation accuracy. East.-Eur. J. Enterp. Technol. 2018, 1(4(91)), 76–82. http://doi.org/10.15587/1729-4061.2018.123921
18. Tymchuk, S., Piskarev, O., Miroshnyk, O., Halko, S., Shchur, T. Expansion of the area of practical application of the plc control system with parallel architecture. Informatyka, Automatyka, Pomiary w Gospodarce i Ochronie Srodowiska, 2022, 12(3), 16–19. http://doi.org/10.35784/iapgos.2983
19. Havrylenko, Y.; Kholodniak, Y.; Halko, S.; Vershkov, O.; Miroshnyk, O.; Suprun, O.; Dereza, O.; Shchur, T.; Śrutek, M. Representation of a monotone curve by a contour with regular change in curvature. Entropy 2021, 23(7), 923. http://doi.org/10.3390/e23070923
20. Kholodniak, Y.; Havrylenko, Y.; Halko, S.; Hnatushenko, V.; Suprun, O.; Volina, T.; Miroshnyk, O.; Shchur, T. Improvement of the algorithm for setting the characteristics of interpolation monotone curve. Informatyka, Automatyka, Pomiary W Gospodarce I Ochronie Środowiska 2023, 13(4), 44–50. https://doi.org/10.35784/iapgos.5392
21. Robbin, J.W.; Salomon, D.A. Introduction to Differential Geometry. Springer Spektrum: Zürich, Switzerland, 2022; 432.
22. Shen, W.; Wang, G.; Huang, F. Direction monotonicity of a rational Bézier curve. Appl. Math. 2016, 31, 1–20. https://doi.org/10.1007/s11766-016-3399-7