Advances in Science and Technology Research Journal

Advances in Science and Technology Research Journal, 2025, 19(6), 373–390
https://doi.org/10.12913/22998624/202982
ISSN 2299-8624, License CC-BY 4.0

Received: 2025.03.12
Accepted: 2025.04.15
Published: 2025.05.01

A new formula for the critical moment of lateral torsional buckling of beams elastically restrained at the support nodes

Rafał Piotrowski[1*], Andrzej Szychowski[1]

[1] Faculty of Civil Engineering and Architecture, Kielce University of Technology, al. Tysiąclecia Państwa Polskiego 7, 25-314 Kielce, Poland

* Corresponding author's e-mail: rafalp@tu.kielce.pl

ABSTRACT

This paper deals with the development of a new formula for the critical moment (M_{cr}) of lateral torsional buckling of a beam for extreme conditions of its support for bending M_y (i.e. simply support or complete restraint) and simultaneous occurrence of elastic restraint against warping (κ_ω) and against lateral rotation (κ_u) at the supports. For specific parameters (κ_ω, κ_u) defining the boundary conditions of lateral torsional buckling of the beam, the full and mutually independent range of elastic restraint degrees was taken into account, from their complete freedom ($\kappa_\omega = 0$, $\kappa_u = 0$) to their complete fixity ($\kappa_\omega = 1$, $\kappa_u = 1$). Single-span beams with a hot-rolled bisymmetric I section (or its welded equivalent) were considered. The most common beam loading schemes encountered in engineering practice were taken into account. In constructing the new approximation formula, the concept of the integrated interaction coefficient was used according to the idea presented in the previous work of the authors. Small percentage differences of the estimated $M_{cr}(\kappa_\omega, \kappa_u)$ values, compared to finite element method (*LTBeamN*), allow us to conclude that the proposed new formula gives sufficiently accurate results. The algorithm used in this paper to create approximation formulas may allow for taking into account a larger number of degrees of beam elastic restraint. The subject of further research will be the interactive consideration of three technically important conditions of elastic beam restraint, i.e. elastic restraint of warping, elastic restraint of lateral rotation (rotation in the lateral torsional buckling plane) and elastic restraint of rotation relative to the major axis of the cross-section (rotation in the bending plane).

Keywords: critical moment of lateral torsional buckling, elastic restraint against warping, elastic restraint against lateral rotation, approximation formulas.

INTRODUCTION

The development of modern methods of designing metal structures tends to more and more accurately represent the actual behaviour of the structural element in the engineering computational model. In this case, numerical simulations using the finite element method (FEM) as well as analytical calculations and approximation formulas are used. The developed formulas are used to verify numerical simulations, and in many technically important cases can be used for basic design. Currently, there is a trend in theoretical research aimed at developing exact or approximate formulas used to verify FEM calculations, e.g. [1–5]. This approach improves the safety of the structure already at the design stage.

One of the important research directions in the field of design of bending elements is to take into account the actual support conditions of beams sensitive to lateral torsional buckling (LTB). So far, the commonly accepted scheme of the so-called "fork" support is in many cases inadequate. Such beams are found, for example, in frame structures, flat or spatial frames and in grillages.

An important design parameter of the transversely loaded steel beams is the elastic critical resistance due to the LTB condition. A convenient measure for design calculations of this resistance is the elastic critical moment (M_{cr}). On the basis of M_{cr}, the so-called relative slenderness ($\overline{\lambda}_{LT}$) and the reduction factor (χ_{LT}) influencing on the design resistance of the beam from the lateral torsional buckling condition are determined. This way of taking into account the phenomenon of instability and the impact of inevitable imperfections is widely accepted in European standards [6, 7] for the design of steel structures.

The M_{cr} elastic critical moment of the beam depends, among others, on: a) the type and geometry of the cross-section, b) the beam loading method (the M_y moment distribution), c) the ordinate of application of the transverse load at the cross-section height, and d) the method of beam restraint at support nodes. A concise list of articles describing the influence of selected factors on the critical moment of lateral torsional buckling of a beam can be found, among others, in [8–10].

In the previous works of the authors [9, 11, 12] the influence of different configurations of elastic restraints in the beam support nodes on its elastic critical moment resulting from the LTB condition was analyzed. In the study [11], only the influence of elastic restraint against warping of beam cross-section at the support nodes was taken into account. The tests considered simply supported beams in bending both about the minor (*z-z*) and major (*y-y*) axis of the cross-section. In this case, the original "coupling" of a certain group of power polynomials was used to approximate the torsion angle function (φ). To determine the M_{cr}, the energy method [13] in the *Rayleigh-Ritz* formulation was used. Based on symbolic "computations", an approximation formula for M_{cr} was derived. The results of the analytical solution [11] were confirmed using FEM (*LTBeam*, *Abaqus*) and the formulas available in [14].

In the paper [12] the interaction of elastic restraint against warping and elastic restraint against rotation in the plane of lateral torsional buckling (lateral rotation) was considered. The studies concerned single-span and simply supported beams at bending M_y, in which independent of each other elastic fixity indexes against warping and against lateral rotation (i.e. concerning the *z-z* axis) were used. In the construction of an analytical model based on the energy method [13], the polynomial "coupling" proposed in the work [11] was used both for the approximation of the torsion angle function (φ) as well as for the approximation of the lateral deflection function (u). Efficient computational programs were developed, obtaining good agreement of results compared to *LTBeam* (FEM). Moreover, based on symbolic "computations" for the first terms of the polynomial "series" (φ and u) approximation formulas were derived. A good approximation of M_{cr} was obtained compared to the values obtained from *LTBeam*. However, the obtained formulas have a much more extensive form compared to the formulas derived in the work [11]. This is due to the use of two displacement functions (φ, u), which extend the function of the total potential energy of the system: beam with elastic restraints – load.

In turn, the study [9] considered the interaction of elastic restraint against warping and elastic restraint against rotation in the bending plane M_y. Such beams occur, for example, in flat frames. The elastic restraint against warping was taken into account in the same way as in [11,12]. However, the elastic restraint against rotation in the bending plane of the beam was taken into account in the form of a properly calibrated the coefficient of interaction. This approach proved to be very effective and allowed for the determination of approximate formulas for M_{cr}, despite the occurrence of the phenomenon of change of the position of the maximum moment as a function of the elastic index of fixity. Also, in this case, good compliance of the designated M_{cr} was obtained compared to FEM (*LTBeamN*).

In this paper, a new formula for M_{cr} from the LTB condition of beams for two extremely different support conditions in the bending plane M_y (i.e. simply support and bilaterally fixed) is developed, assuming their elastic fixing against warping and lateral rotation. Single-span beams with a hot-rolled bisymmetric I section (or its welded equivalent) were considered. For the above-mentioned nodal fixing parameters, the full and mutually independent range of variation of the degree of elastic restraint was taken into account, from complete freedom of warping to its complete prevent and from complete freedom of lateral rotation to its complete prevent. In the construction of approximation formulas, the concept of integrated interaction coefficient was used according to the idea presented in the paper [9], which significantly simplified the calculations.

Taking into account a total of three technically important conditions of elastic fixation (i.e. taking into account also the stiffness of elastic fixation to bending M_y) will be the subject of further research by the authors.

The following assumptions were made in this work: (1) the single-span beams have a constant, bisymmetrical hot-rolled I section or its welded equivalent, (2) the same boundary conditions occur at both support nodes, (3) the three most typical and most frequently encountered load schemes in engineering practice were taken into account, (4) the conditions of elastic beam fixation in support sections include one rotational (relative to the minor axis of the cross-section) and one warping degree of freedom, with the simultaneous occurrence of one of the two extreme boundary conditions for bending M_y (simply support or full restraint).

Compared to the solutions available in the literature, the paper offers an innovative approach that takes into account:

- derivation of a new and simpler formulation of the approximation formula for M_{cr} (with integrated interaction coefficient), while taking into account any degree of elastic restraint of the stiffness of the nodes against: warping and rotation about the minor axis of the cross-section for two extreme support conditions relative to the major axis of the cross-section (simply support, full restraint),
- obtaining an analytical solution that allows for relatively simple and sufficiently accurate, from a technical point of view, consideration of the actual behaviour of a steel beam sensitive to LTB, which is part of a frame structure, e.g. a grillage or frame,
- obtaining a solution constituting the starting point and reference point for deriving approximation formulas on M_{cr} taking into account the elastic impact of three, technically important from the point of view of the design of frame structures, elastic restraints, i.e. restraints against: a) warping, b) lateral rotation, c) rotation in the bending plane M_y.

Caution: The approximation formula derived in the work [12] on M_{cr} applies only to beams simply supported in the plane of the main bending M_y and has a much more extensive form compared to the proposal presented in this work. In addition, the methodology of using the integrated interaction coefficient adopted in this paper allows for the mathematical unification of the proposed formulas, which can potentially allow their extension to even more complex conditions of elastic fixation in nodes.

BEAM FIXING CONDITIONS AT SUPPORT NODES

An example of a static scheme of a single-span beam fully restrained in bending (M_y) about the major axis of the cross-section and loaded with a concentrated force at the mid-span is shown in Figure 1a. The analysis takes into account the elastic stiffness of the node (Fig. 1b) against: a) warping α_ω (red) and b) rotation in the lateral torsional buckling plane α_u (blue). Two extreme boundary conditions for bending relative to the axis of higher stiffness α_v (simply support α_v = 0 or full fixation $\alpha_v = \infty$) were also taken into account (Fig. 1c).

The elastic stiffnesses α_ω [9, 11, 12, 15, 16] and α_u [12], express the values of generalized cross-sectional forces (i.e. bimoment B and moment M_z) caused by unit generalized displacements (warping $d\varphi/dx$ and lateral rotation

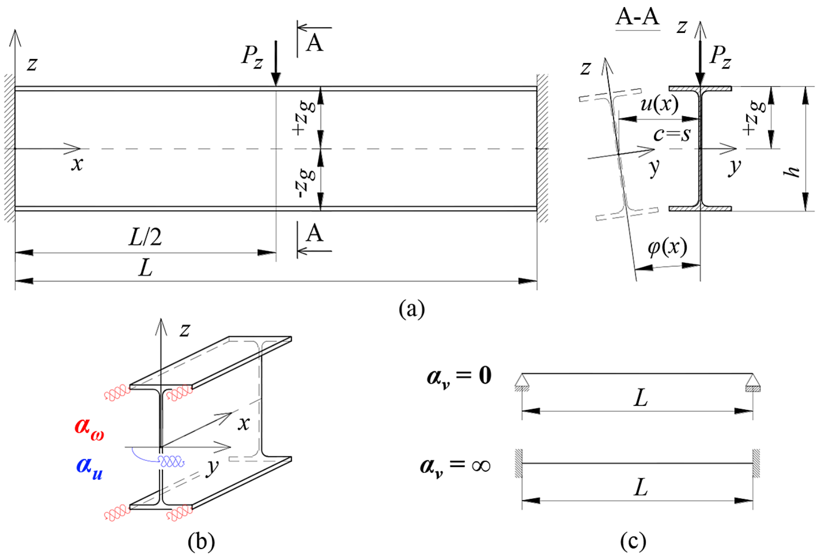


Figure 1. (a) Static scheme of a beam loaded with concentrated force, (b) Elastic stiffnesses at the support node, (c) Conditions for fixing the beam in the bending plane

du/*dx*, respectively). The stiffnesses (α_ω, α_u) varies from $\alpha_\omega = 0$, $\alpha_u = 0$ in complete absence of stiffness ("fork" support) to $\alpha_\omega = \infty$, $\alpha_u = \infty$ in complete fixation.

The elastic fixities of beam are taken into account by means of dimensionless elastic restraint indexes κ_ω [9, 11, 12] and κ_u [12].

The dimensionless index of the elastic fixation against warping κ_ω was determined on the basis of α_ω in the form [9, 11, 12] of:

$$\kappa_\omega = \frac{\alpha_\omega L}{2EI_\omega + \alpha_\omega L} \tag{1}$$

where: L – beam span, E – Young's modulus, I_ω – warping constant, and α_ω – rigidity of elastic restraint against warping [9,11,12,15,16].

The index of elastic fixity against warping changes is from $\kappa_\omega = 0$ for complete warping freedom to $\kappa_\omega = 1$ for full prevention of warping.

The dimensionless index of the elastic fixation against lateral rotation (i.e. in the LTB plane) κ_u was determined on the basis of α_u as [12]:

$$\kappa_u = \frac{\alpha_u L}{2EI_z + \alpha_u L} \tag{2}$$

where: I_z – second moment of inertia in bending about the *z*-axis, and α_u – rigidity of elastic restraint against lateral rotation [12].

The index of elastic fixity against lateral rotation changes is from $\kappa_u = 0$ for complete lateral rotation freedom to $\kappa_u = 1$ for full prevention of lateral rotation.

The simple transformation of formulas (1) and (2) allows the expression of stiffnesses (α_ω, α_u) in the function of indexes (κ_ω, κ_u) according to formulas [12]:

$$\alpha_\omega = \frac{2\kappa_\omega EI_\omega}{(1-\kappa_\omega)L} \qquad \alpha_u = \frac{2\kappa_u EI_z}{(1-\kappa_u)L} \tag{3}$$

The static schemes of the beam adopted in the analysis (Fig. 1c) take into account two extreme conditions of its fixing in the M_y moment bending plane (simply support $\alpha_v = 0$, full fixation $\alpha_v = \infty$). According to the [9] markings, the simply support is described with the index $\kappa_v = 0$ and the full fixation with the index $\kappa_v = 1$.

THE PREVIOUS APPROXIMATION FORMULA FOR THE M_{cr}

The approximation formula for the critical moment M_{cr} LTB of a beam simply supported in the bending plane ($\kappa_v = 0$), taking into account the elastic restraint against warping ($0 \le \kappa_\omega \le 1$) and the elastic restraint against lateral rotation ($0 \le \kappa_u \le 1$) was determined in [12]:

$$M_{cr} = D_1 \frac{-B_1 EI_z z_g + \sqrt{EI_z\left(B_2 GI_t L^2 + B_3 EI_\omega + B_1^2 EI_z z_g^2\right)}}{B_4 L^2} \tag{4}$$

where: B_1, B_2, B_3, B_4, D_1 – coefficients according to Table 1, z_g – ordinate of the point of transverse load application with respect to shear centre (see Fig. 1a), G – Kirchhoff's modulus, I_t – Saint-Venant torsional constant.

Table 1 lists the B_1, B_2, B_3, B_4 and D_1 coefficients for beams simply supported against bending ($\kappa_v = 0$) and the most common loading schemes.

As already mentioned in the introduction, the approximation formula (4) proposed in [12] allows for obtaining a good approximation of M_{cr} compared to the values obtained from the *LT-Beam* software. It should be noted, however, that the obtained approximation formulas (see Table 1) have a very extended form. Moreover, the solution proposed in [12] is dedicated only for simply supported beams. For this reason, in the next chapter of the paper a new, simplified formula for $M_{cr}(\kappa_\omega, \kappa_u)$ for simply supported ($\kappa_v = 0$) or bilaterally fixed ($\kappa_v = 1$) beams is proposed.

A NEW FORM OF THE APPROXIMATION FORMULA FOR THE M_{cr} OF BEAMS ELASTICALLY RESTRAINED AGAINST WARPING (κ_ω) AND AGAINST LATERAL ROTATION (κ_u)

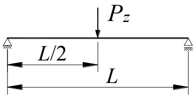
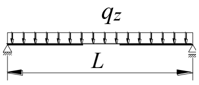
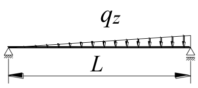
To determine the M_{cr} of the beams elastically restrained against warping (κ_ω) and against lateral rotation (κ_u), the idea of an integrated interaction coefficient presented in the works [2,9] was used.

The approximation formula for M_{cr} can be presented as:

$$\begin{aligned} M_{cr}(\kappa_\omega,\kappa_u) &= M_{cr,o}(\kappa_\omega,\kappa_u = 0) + \\ &+ \left[M_{cr,u}(\kappa_\omega,\kappa_u = 1) - M_{cr,o}(\kappa_\omega,\kappa_u = 0)\right]\cdot\eta(\kappa_u) \end{aligned} \tag{5}$$

where: $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ – the LTB critical moment for beam with complete freedom of lateral rotation and a given value of the κ_ω index, $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$ – the LTB critical moment for beam with complete blockage of lateral rotation and a given value of the κ_ω index, $\eta(\kappa_u)$ – the coefficient of interaction determined for a given value of the $0 < \kappa_u < 1$ index.

Table 1. Coefficients B_1, B_2, B_3, B_4 and D_1 for simply supported beams ($\kappa_v = 0$) for M_{cr} [12]

Item	Static scheme	Coefficients
I	II	III
1		$B_1 = 11.52 \cdot (1.2 - \kappa_u) \cdot (1.563 - 2.5\kappa_\omega + \kappa_\omega^2)$ $B_2 = 30.719 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_3 = 368.638 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.2 - \kappa_\omega)$ $B_4 = \alpha_1 - \alpha_2\kappa_\omega + \alpha_3\kappa_\omega^2$ $D_1 = \beta_1 + \beta_2 z_g$ auxiliary coefficients: $\alpha_1 = 1.488 \cdot (1.761 - 2.654\kappa_u + \kappa_u^2)$; $\alpha_2 = 2.44 \cdot (1.752 - 2.647\kappa_u + \kappa_u^2)$ $\alpha_3 = 1.742 - 2.64\kappa_u + \kappa_u^2$ $\beta_1 = 0.043 \cdot (21.855 - \kappa_u) + 0.004 \cdot (-1.399 + \kappa_u^5) \cdot \kappa_\omega + 0.081 \cdot (-0.398 - \kappa_u^3) \cdot \kappa_\omega^5$ $\beta_2 = 0.001 \cdot (0.42 + \kappa_u) + 0.0001 \cdot (0.182 - \kappa_u^2) \cdot \kappa_\omega + 0.001 \cdot (0.327 + \kappa_u^5) \cdot \kappa_\omega^5$
2		$B_1 = 7.5 \cdot (1.2 - \kappa_u) \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 18.749 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_3 = 225.001 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.2 - \kappa_\omega)$ $B_4 = \alpha_1 - \alpha_2\kappa_\omega + \alpha_3\kappa_\omega^2$ $D_1 = \beta_1 + \beta_2 z_g$ auxiliary coefficients: $\alpha_1 = 1.474 \cdot (1.675 - 2.588\kappa_u + \kappa_u^2)$; $\alpha_2 = 2.429 \cdot (1.664 - 2.58\kappa_u + \kappa_u^2)$ $\alpha_3 = 1.653 - 2.571\kappa_u + \kappa_u^2$ $\beta_1 = 0.029 \cdot (33.337 - \kappa_u) + 0.006 \cdot (-0.432 - \kappa_u) \cdot \kappa_\omega + 0.103 \cdot (-0.293 + \kappa_u^5) \cdot \kappa_\omega^5$ $\beta_2 = 0.001 \cdot (0.246 + \kappa_u^2) + 0.0002 \cdot (0.015 - \kappa_u^5) \cdot \kappa_\omega + 0.001 \cdot (-0.002 - \kappa_u^5) \cdot \kappa_\omega^5$
3		$B_1 = 7.68 \cdot (1.2 - \kappa_u) \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 19.66 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_3 = 235.929 \cdot B_4 \cdot (1.2 - \kappa_u) \cdot (1.2 - \kappa_\omega)$ $B_4 = \alpha_1 - \alpha_2\kappa_\omega + \alpha_3\kappa_\omega^2$ $D_1 = \beta_1 + \beta_2 z_g$ auxiliary coefficients: $\alpha_1 = 1.474 \cdot (1.675 - 2.588\kappa_u + \kappa_u^2)$; $\alpha_2 = 2.429 \cdot (1.664 - 2.58\kappa_u + \kappa_u^2)$ $\alpha_3 = 1.653 - 2.571\kappa_u + \kappa_u^2$ $\beta_1 = 0.033 \cdot (29.373 - \kappa_u) + 0.005 \cdot (-0.512 - \kappa_u) \cdot \kappa_\omega + 0.103 \cdot (-0.279 + \kappa_u^5) \cdot \kappa_\omega^5$ $\beta_2 = 0.001 \cdot (0.174 + \kappa_u^2) + 0.0002 \cdot (-0.054 - \kappa_u^5) \cdot \kappa_\omega + 0.001 \cdot (0.024 - \kappa_u^5) \cdot \kappa_\omega^5$

The values of the $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ and $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$ moments for the extreme conditions of beam attachment in the plane of its bending should be calculated using the formulas proposed in the next chapter of this work.

Equation 5 can be converted to a simplified form:

$$M_{cr} = M_o + (M_u - M_o) \cdot \eta \qquad (6)$$

where: $M_o = M_{cr,o}(\kappa_\omega, \kappa_u = 0)$, $M_u = M_{cr,u}(\kappa_\omega, \kappa_u = 1)$, $\eta = \eta(\kappa_u)$.

Using formulas (5) and (6), $M_{cr}(\kappa_\omega, \kappa_u)$ can be determined for extreme conditions of beam fixing in the bending plane (M_y), i.e. for simply support ($\kappa_v = 0$) and full fixation ($\kappa_v = 1$), which significantly facilitates the calculations. For this purpose, it is necessary to calculate the moments (M_o and M_u) in the function of κ_ω and determine the interaction coefficient η in the function of κ_u.

This approach facilitates the construction of approximation formulas, as it is based on the extreme values M_o and M_u of the critical moment, which are usually determined much more simply than for the "double" elastic restraint conditions (i.e. for two elastic fixings at the same time) and on the integrated interaction coefficient η, the form of which can be determined in a manner analogous to that in [9].

APPROXIMATION FORMULAS FOR $M_{cr}(K_\omega)$ FOR EXTREME BOUNDARY CONDITIONS (K_u, K_v)

M_{cr} of beams elastically restrained against warping (κ_ω), simply supported for lateral rotation ($\kappa_u = 0$) and simply supported in the bending plane ($\kappa_v = 0$)

In the case of a beam simply supported both against lateral rotation about the minor axis of the cross-section ($\kappa_u = 0$) and in the plane of the major bending ($\kappa_v = 0$) and elastically restrained against warping ($0 \leq \kappa_\omega \leq 1$) in the support nodes, the critical moment of lateral torsional buckling can be determined using the formula (7) given in [11]:

$$M_{cr,o}(\kappa_\omega, \kappa_u = 0) = \frac{-B_1 EI_z z_g + \sqrt{EI_z\left(B_3 GI_t L^2 + B_4 EI_\omega + B_1^2 EI_z z_g^2\right)}}{B_2 L^2} \quad (7)$$

where: B_1, B_2, B_3, B_4 – coefficients according to Table 2 and z_g – ordinate of the point of transverse load application with respect to shear centre (Fig. 1a).

Table 2 lists the B_1, B_2, B_3 and B_4 coefficients for beams simply supported against bending (κ_v = 0), with complete freedom of lateral rotation (κ_u = 0) and the most common loading schemes [11].

M_{cr} of beams elastically restrained against warping (κ_ω), simply supported for lateral rotation (κ_u = 0) and fully fixed in the bending plane (κ_v = 1)

The approximation formula for the critical moment M_{cr} LTB of the beam, fully restrained in the bending plane (κ_v = 1), taking into account the simply support on lateral rotation (κ_u = 0) and the elastic restraint against warping ($0 \le \kappa_\omega \le 1$) was determined in the work [9]. The formula has the form (7), and the values of coefficients B_1, B_2, B_3 and B_4 are given in Table 3 [9].

M_{cr} of beams elastically restrained against warping (κ_ω), fully restrained against lateral rotation (κ_u = 1) and simply supported in the bending plane (κ_v = 0)

In the case of a beam simply supported (κ_v = 0) in the bending plane, with a full lateral rotation restrained (κ_u = 1) and the elastic restraint against warping ($0 \le \kappa_\omega \le 1$) in the support nodes, the proposed formula is as follows (8):

$$M_{cr,u}(\kappa_\omega, \kappa_u = 1) = D_1 \frac{-B_1 EI_z z_g + \sqrt{EI_z\left(B_3 GI_t L^2 + B_4 EI_\omega + B_1^2 EI_z z_g^2\right)}}{B_2 L^2} \quad (8)$$

where: B_1, B_2, B_3, B_4, D_1 – coefficients according to Table 4 and z_g – ordinate of the point of transverse load application with respect to shear centre (Fig. 1a).

Table 2. Coefficients B_1, B_2, B_3 and B_4 for simply supported beams (κ_v = 0) for $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ [11]

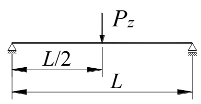
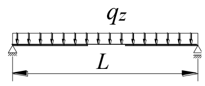
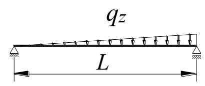
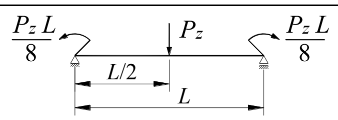
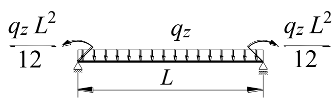
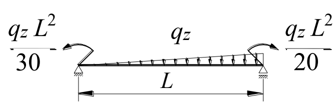
Item	Static scheme	Coefficients
I	II	III
1		$B_1 = 7.242 \cdot (1.563 - 2.5\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.522 - 2.467\kappa_\omega + \kappa_\omega^2$ $B_3 = 19.248 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 231.816 \cdot B_2 \cdot (1.2 - \kappa_\omega)$
2		$B_1 = 5.25 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.507 - 2.455\kappa_\omega + \kappa_\omega^2$ $B_3 = 13.092 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 157.633 \cdot B_2 \cdot (1.2 - \kappa_\omega)$
3		$B_1 = 5.322 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.507 - 2.455\kappa_\omega + \kappa_\omega^2$ $B_3 = 13.624 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 163.486 \cdot B_2 \cdot (1.2 - \kappa_\omega)$

Table 3. Coefficients B_1, B_2, B_3 and B_4 for bilaterally fixed beams (κ_v = 1) for $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ [9]

Item	Static scheme	Coefficients
I	II	III
1		$B_1 = 23.333 \cdot (1.563 - 2.5\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.522 - 2.467\kappa_\omega + \kappa_\omega^2$ $B_3 = 31.032 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 372.934 \cdot B_2 \cdot (1.2 - \kappa_\omega)$
2		$B_1 = 42 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.507 - 2.455\kappa_\omega + \kappa_\omega^2$ $B_3 = 69.692 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 839.664 \cdot B_2 \cdot (1.2 - \kappa_\omega)$
3		$B_1 = 49.033 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.507 - 2.455\kappa_\omega + \kappa_\omega^2$ $B_3 = 102.445 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 1234.274 \cdot B_2 \cdot (1.2 - \kappa_\omega)$

To derive the formula (8), the energy method [13] was used in a manner analogous to the procedure described in [11]. The behaviour of the beam in the lateral torsional buckling phase is described by the first term of the torsion angle function φ_1 and the first term of the lateral deflection function u_1 according to [11]. In the case of the torsion angle function, the full (from 0 to 1) variation of the κ_ω index was taken into account, and for the lateral deflection function, the full rotation restraint was assumed, i.e. $\kappa_u = 1$. This approach made symbolic calculations much easier than the procedure described in [12]. The formulas for the coefficients B_1 to B_4 and D_1 derived in this paper are presented in Table 4.

Table 4 lists the B_1, B_2, B_3, B_4 and D_1 coefficients for beams simply supported against bending ($\kappa_v = 0$), with complete blockage of lateral rotation ($\kappa_u = 1$) and the most common loading schemes.

M_{cr} of beams elastically restrained against warping (κ_ω), fully restrained against lateral rotation ($\kappa_u = 1$) and fully fixed in the bending plane ($\kappa_v = 1$)

To calculate the critical lateral torsional buckling moment M_{cr} of a beam, taking into account the full rotation restraint in the bending plane ($\kappa_v = 1$) and in the LTB plane ($\kappa_u = 1$) with the elastic restraint of warping (κ_ω), formula (8) can be used.

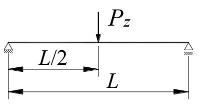
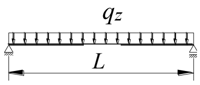
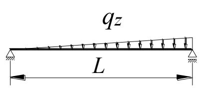
The form of formula (8) and the functions of coefficients B_1, B_2, B_3, B_4 and D_1 were determined in a similar way as for those presented in the previous subsection of the paper. Differences in the value of some components result from a different form of the function of external forces. The formulas for coefficients B_1 to B_4 and D_1 derived in this paper, for $\kappa_v = 1$, are given in Table 5.

INTEGRATED INTERACTION COEFFICIENT

To develop and calibrate the formulas for the integrated interaction coefficient $\eta(\kappa_u)$, the relationships of M_{cr} in the function of elastic restraint indexes (κ_ω, κ_u, κ_v) determined using FEM simulation (*LTBeamN*) for a selected group of hot-rolled I-beams (IPE, HEB, HEA) were analyzed. *LTBeamN* [17] is a well-known engineering software that uses the finite element method. The computational algorithm is based on the so-called thin-walled bar finite elements (7 degrees of freedom at the node). The software allows to determine the M_{cr} of a beam taking into account, among others, the complex conditions of its elastic restraints in the support nodes. *LTBeamN* is successfully used, among others, in the verification of analytical solutions and approximation formulas proposed in various works [18–21].

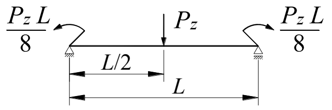
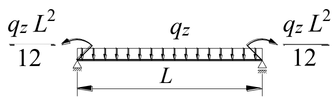
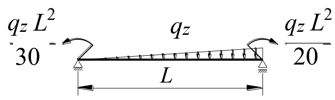
Figures 2, 3 and 4 show example curves of M_{cr} variation as a function of the κ_u index for beams: 1) IPE300 with span $L = 5$ m for $\kappa_\omega = 0.6$ (Fig. 2), 2) HEA400 with span $L = 7$ m for $\kappa_\omega = 0.4$ (Fig. 3) and, 3) HEB500 with span $L = 10$ m for $\kappa_\omega = 0.8$ (Fig. 4). The beams were simply supported in the

Table 4. Coefficients B_1, B_2, B_3, B_4 and D_1 for simply supported beams ($\kappa_v = 0$) for $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$

Item	Static scheme	Coefficients
I	II	III
1		$B_1 = 22.5 \cdot (1.563 - 2.5\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.554 - 2.493\kappa_\omega + \kappa_\omega^2$ $B_3 = 60 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 720 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.92 + 0.07 \cdot \frac{z_g}{h} - 0.03\kappa_\omega$
2		$B_1 = 18.375 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.563 - 2.5\kappa_\omega + \kappa_\omega^2$ $B_3 = 45.937 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 551.25 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.96 + 0.07 \cdot \frac{z_g}{h} - 0.03\kappa_\omega$
3		$B_1 = 18.816 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.563 - 2.5\kappa_\omega + \kappa_\omega^2$ $B_3 = 48.169 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 578.028 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.96 + 0.07 \cdot \frac{z_g}{h} - 0.03\kappa_\omega$

Note: z_g – ordinate of the point of transverse load application with respect to shear centre (see Fig. 1a), h – high of the beam (see Fig. 1a).

Table 5. Coefficients B_1, B_2, B_3, B_4 and D_1 for bilaterally fixed beams ($\kappa_v = 1$) for $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$

Item	Static scheme	Coefficients
I	II	III
1		$B_1 = 45 \cdot (1.563 - 2.5\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.458 - 2.415\kappa_\omega + \kappa_\omega^2$ $B_3 = 60 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 720 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.8 + 0.3 \cdot \frac{z_g}{h} - 0.05\kappa_\omega$
2		$B_1 = 70.56 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.44 - 2.4\kappa_\omega + \kappa_\omega^2$ $B_3 = 117.6 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 1411.2 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.9 + 0.22 \cdot \frac{z_g}{h} - 0.05\kappa_\omega$
3		$B_1 = 84.672 \cdot (1.476 - 2.429\kappa_\omega + \kappa_\omega^2)$ $B_2 = 1.44 - 2.4\kappa_\omega + \kappa_\omega^2$ $B_3 = 169.344 \cdot B_2 \cdot (1.457 - 2.4\kappa_\omega + \kappa_\omega^2)$ $B_4 = 2032.128 \cdot B_2 \cdot (1.2 - \kappa_\omega)$ $D_1 = 0.9 + 0.22 \cdot \frac{z_g}{h} - 0.05\kappa_\omega$

Note: z_g – ordinate of the point of transverse load application with respect to shear centre (see Fig. 1a), h – high of the beam (see Fig. 1a).

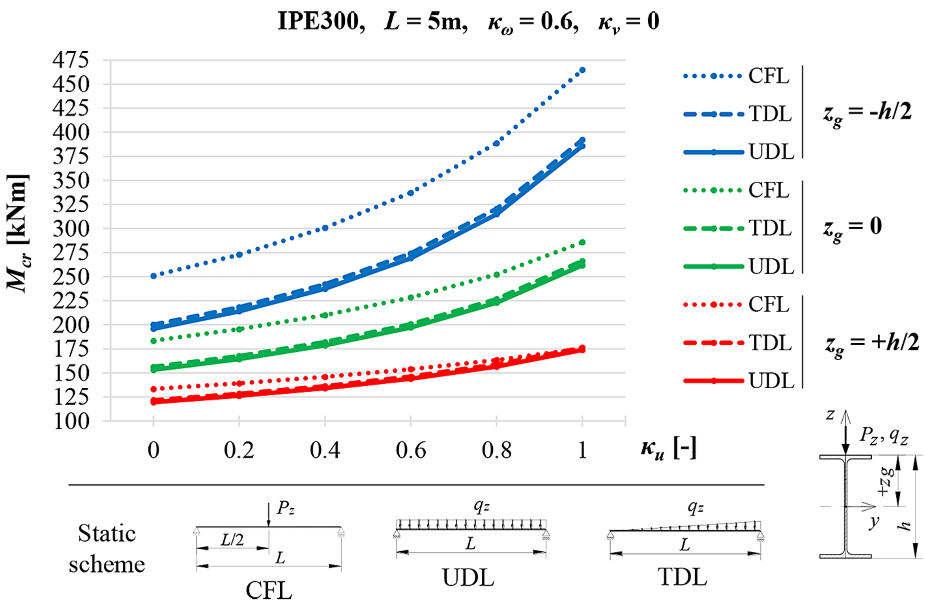


Figure 2. Trends in $M_{cr}(\kappa_u)$ variation according to *LTBeamN* software for simply supported ($\kappa_v = 0$) IPE300 beam ($L = 5$ m)

bending plane ($\kappa_v = 0$) and loaded successively: a) concentrated force load (CFL) at the centre of the span, b) uniform distributed load (UDL), c) triangle distributed load (TDL).

The analysis of Figures 2, 3 and 4 shows that the highest trend of non-linear increase of M_{cr} in function κ_u was observed for loads applied to the bottom flange of the beam (blue line). As the ordinate of load application increases, the nonlinear increase of M_{cr} decreases. Similar relationships were observed for the entire range of variability of the index $\kappa_\omega = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ in individual cross-section types.

Due to the observed nonlinear increases of $M_{cr}(\kappa_u)$ for simply supported beams ($\kappa_v = 0$), the interaction coefficient $\eta(\kappa_u)$ was described by the second-degree function (Table 6). For its calibration, M_{cr} obtained from FEM (*LTBeamN*) was used, for the IPE300 beam with span $L = 5$m, for restraint index $\kappa_v = 0$, and for the full range of indexes $\kappa_\omega = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ and $\kappa_u = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

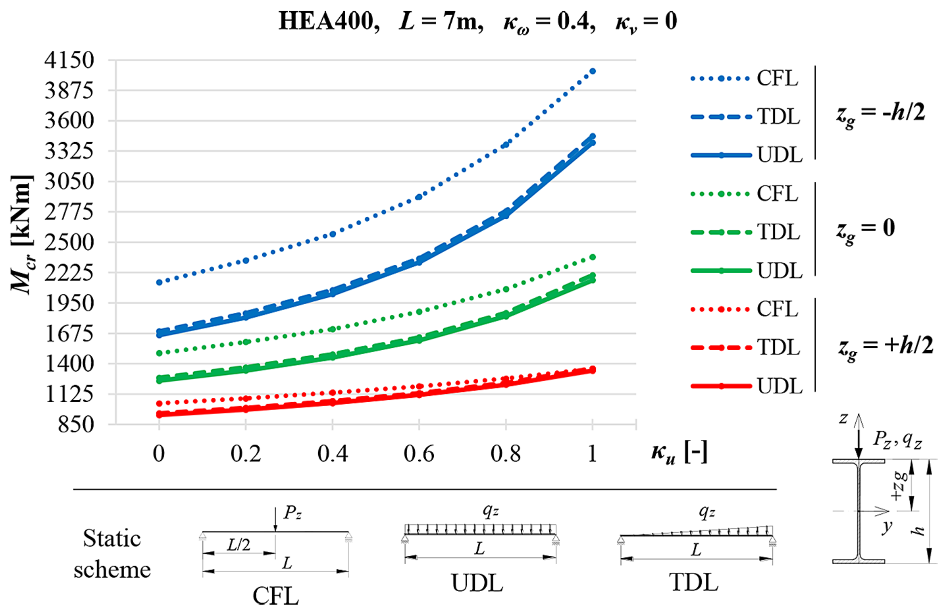


Figure 3. Trends in $M_{cr}(\kappa_u)$ variation according to *LTBeamN* software for simply supported ($\kappa_v = 0$) HEA400 beam (L = 7 m)

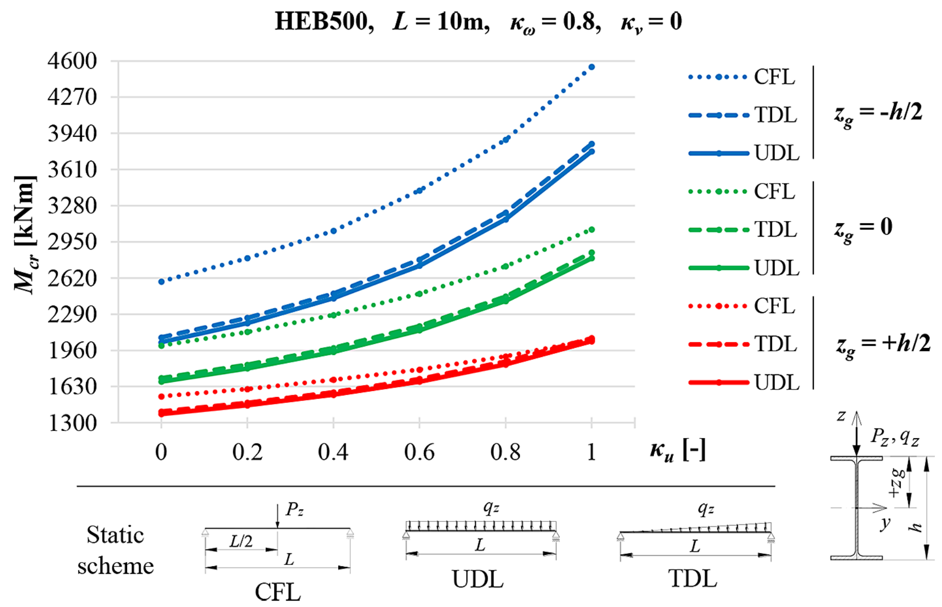
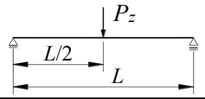
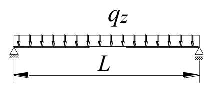
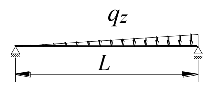


Figure 4. Trends in $M_{cr}(\kappa_u)$ variation according to *LTBeamN* software for simply supported ($\kappa_v = 0$) HEB500 beam (L = 10 m)

In practical calculations of the critical moment for simply supported beams in bending M_y ($\kappa_v = 0$) using the approximation formula (5) and the integrated interaction coefficient according to Table 6, it should be assumed that the calculations are carried out for the range κ_u from 0.1 to 0.9. In the extreme ranges, i.e. for κ_u from 0 to 0.1 and from 0.9 to 1, the M_{cr} values should be interpolated linearly, i.e. between the value $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ from formula (7) and $M_{cr}(\kappa_\omega, \kappa_u = 0.1)$ from formula (5) for the lower range, and between the value $M_{cr}(\kappa_\omega, \kappa_u = 0.9)$ from formula (5) and $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$ from formula (8) for the upper range. This assumption made it possible to simplify the function of the interaction coefficient $\eta(\kappa_u)$, whose form for the full range of κ_u (0 ÷ 1) was excessively complicated.

Figure 5 shows example $M_{cr}(\kappa_u)$ variation curves for an IPE300 beam with a span of L = 5m, full restrained in the bending plane ($\kappa_v = 1$). The load parameters and the restraint index κ_ω were assumed in the same way as in Figure 2.

Table 6. Coefficient of interaction $\eta(\kappa_u)$ for simply supported beams ($\kappa_v = 0$) for $M_{cr}(\kappa_\omega, \kappa_u)$

Item	Static scheme	Coefficient
I	II	III
1		$\eta(\kappa_u) = \left(0.66 - 0.17\frac{z_g}{h}\right)\kappa_u^2 + \left(0.27 + 0.25\frac{z_g}{h}\right)\kappa_u - 0.02\frac{z_g}{h} + 0.01$
2		
3		

Note: z_g – ordinate of the point of transverse load application with respect to shear centre (see Fig. 1a), h – high of the beam (see Fig. 1a).

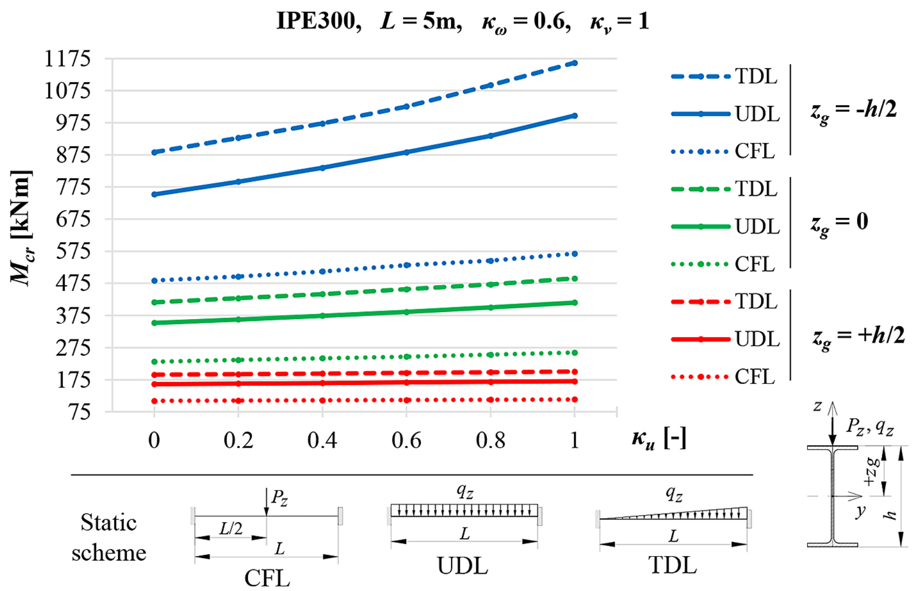


Figure 5. Trends in $M_{cr}(\kappa_u)$ variation according to *LTBeamN* software for bilaterally fixed ($\kappa_v = 1$) IPE300 beam ($L = 5$ m)

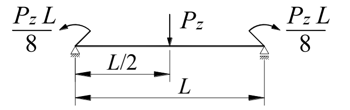
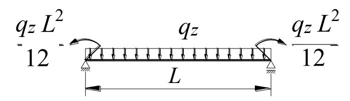
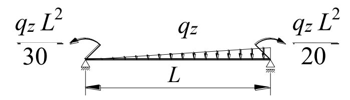
Similarly to Figure 3 and Figure 4, very similar $M_{cr}(\kappa_u)$ variation curves for beams restrained in the bending plane ($\kappa_v = 1$) were also obtained for HEA and HEB type cross-sections.

The analysis of Figure 5 shows that the highest trend, in principle, of linear increase M_{cr} in function κ_u, was noted for loads applied to the bottom beam flange (blue line). With the increase of the ordinate of load application, the linear increase of M_{cr} decreases. Similar relationships were observed for the entire range of variability of the index $\kappa_\omega = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

Due to the observed almost linear increases of $M_{cr}(\kappa_u)$ for beams fully restrained against rotation in the bending plane ($\kappa_v = 1$), the interaction coefficient $\eta(\kappa_u)$ is described by the first-degree function (Table 7). For its calibration, M_{cr} obtained from FEM (*LTBeamN*) was used, for the IPE300 beam with span $L = 5$m, for restraint index $\kappa_v = 1$, and for the full range of indexes $\kappa_\omega = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ and $\kappa_u = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

Similarly to the previous case (for $\kappa_v = 0$), in practical calculations of the critical moment of beams bilaterally fixed at bending M_y ($\kappa_v = 1$), using the approximation formula (5) and the integrated interaction coefficient according to Table 7, it should be assumed that the calculations are carried out for the range κ_u from 0.1 to 0.9. In the extreme ranges, i.e. for κ_u from 0 to 0.1 and from 0.9 to 1, the M_{cr} values should be interpolated linearly, i.e. between the value $M_{cr,o}(\kappa_\omega, \kappa_u = 0)$ from

Table 7. Coefficient of interaction $\eta(\kappa_u)$ for bilaterally fixed beams ($\kappa_v = 1$) for $M_{cr}(\kappa_\omega, \kappa_u)$

Item	Static scheme	Coefficient
I	II	III
1		$\eta(\kappa_u) = \kappa_u + 0.08\frac{z_g}{h} - 0.05$
2		
3		

Note: z_g – ordinate of the point of transverse load application with respect to shear centre (see Fig. 1a), h – high of the beam (see Fig. 1a).

formula (7) and $M_{cr}(\kappa_\omega, \kappa_u = 0.1)$ from formula (5) for the lower range, and between the value $M_{cr}(\kappa_\omega, \kappa_u = 0.9)$ from formula (5) and $M_{cr,u}(\kappa_\omega, \kappa_u = 1)$ from formula (8) for the upper range. This assumption made it possible to simplify the function of the interaction coefficient $\eta(\kappa_u)$, whose form for the full range of κ_u (0 ÷ 1) was excessively complicated.

EXAMPLES

In the comparative examples, the critical moments M_{cr} were estimated for steel beams made of sections: a) IPE300, HEA300, HEB300 for spans $L = 5$ and 7 m, and b) IPE500, HEA500, HEB500 for spans $L = 8$ and 10 m. The static schemes of the beams presented in Tables 6 and 7 were taken into account for the loads applied to: the top flange ($z_g = +h/2$), the gravity axis ($z_g = 0$) and the bottom flange ($z_g = -h/2$) (see Fig. 1a). The calculations were carried out for the entire range of variation of the elastic fixing indexes for the warping κ_ω and lateral rotation κ_u, assuming $\kappa_i = \{0, 0.25, 0.5, 0.75, 0.9, 1\}_{i=\omega,u}$. The calculations were carried out for various combinations of indexes κ_ω and κ_u. In accordance with the assumptions of this stage of research, it was assumed that the analyzed beams were simply supported ($\kappa_v = 0$) or fully fixed ($\kappa_v = 1$) in bending M_y at both support nodes (cf. Tables 6, 7).

Table 8 presents example results of $M_{cr}(\kappa_\omega, \kappa_u)$ calculations for an IPE300 beam uniformly loaded at the height of the top flange ($z_g = +h/2$) with a span of $L = 5$ m in two variants of restraint on M_y: a) simply support (col. IV – VI), b) full fixing (col. VII – IX). The critical moments were estimated using formula (5) and calculated numerically using the *LTBeamN* (FEM) software.

The critical moments of lateral torsional buckling (Table 8) estimated using formula (5) differed (col. VI, IX) from the values determined using the *LTBeamN* (FEM) software in the range from -0.6 to +4.0%. The comparison of the M_{cr} values shows that an increase in the κ_ω index value usually causes a higher increase in the elastic critical moment than in the case of the same increase in the κ_u index. This applies in particular to a beam bilaterally fixed in the plane of main bending M_y.

Table 9 presents example results of $M_{cr}(\kappa_\omega, \kappa_u)$ calculations for selected combinations of κ_ω, κ_u indexes values for simply supported ($\kappa_v = 0$) or bilaterally fixed ($\kappa_v = 1$) beams with IPE300 cross-section ($L = 5$m), loaded with a concentrated force in the middle of the span or a triangle distributed load. The loads were applied at the height of the top flange ($z_g = +h/2$) of the cross-section. M_{cr} was estimated using formula (5) and determined using FEM (*LTBeamN*).

In addition to comparisons, the $M_{cr}(\kappa_\omega, \kappa_u)$ values presented in Tables 8 and 9 can be used to verify the correctness of the notation of formulas (5) to (8) and the formulas of the coefficients from Tables 2 to 7 in spreadsheets.

Caution: The lower values of critical moments observed in some places in Tables 8 and 9 for bilaterally fixed beams ($\kappa_v = 1$) compared to simply supported beams ($\kappa_v = 0$) with the same values of the κ_ω and κ_u indexes result from the longitudinal distribution of the bending moment M_y and, in some cases, the change in the location of the maximum moment identified with M_{cr} (e.g. for a simply supported and uniformly loaded

Table 8. Comparison of $M_{cr}(\kappa_{\omega}, \kappa_u)$

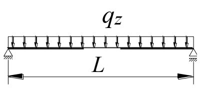
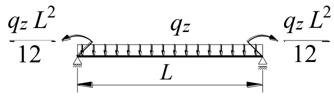
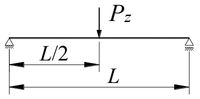
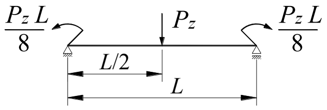
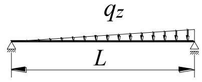
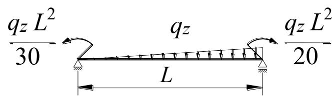
Item	κ_{ω} [-]	κ_u [-]	Static scheme					
			IPE300, L = 5m, z_g = +h/2  κ_v = 0			IPE300, L = 5m, z_g = +h/2  κ_v = 1		
			M_{cr} [kNm]		% [-]	M_{cr} [kNm]		% [-]
			LTBeamN	Formula (5)	V-IV	*LTBeamN*	Formula (5)	VIII-VII
I	II	III	IV	V	VI	VII	VIII	IX
1	0	0	98.74	98.83	+0.1	124.34	124.20	-0.1
2		0.25	105.45	104.91	-0.5	125.51	126.67	+0.9
3		0.5	113.95	114.23	+0.2	126.74	129.25	+2.0
4		0.75	125.26	126.79	+1.2	127.99	131.82	+3.0
5		0.9	134.21	135.88	+1.2	128.75	133.36	+3.6
6		1	141.55	143.94	+1.7	129.28	134.49	+4.0
7	0.25	0	104.62	104.73	+0.1	134.55	134.45	-0.1
8		0.25	111.82	111.15	-0.6	135.92	136.88	+0.7
9		0.5	120.94	121.00	0.0	137.34	139.41	+1.5
10		0.75	133.13	134.28	+0.9	138.81	141.94	+2.3
11		0.9	142.79	143.89	+0.8	139.74	143.46	+2.7
12		1	150.75	152.41	+1.1	140.39	144.58	+3.0
13	0.5	0	114.01	114.16	+0.1	151.34	151.30	0.0
14		0.25	121.98	121.20	-0.6	153.09	153.91	+0.5
15		0.5	132.10	131.98	-0.1	154.87	156.63	+1.1
16		0.75	145.65	146.52	+0.6	156.77	159.34	+1.6
17		0.9	156.42	157.05	+0.4	157.93	160.97	+1.9
18		1	165.29	166.39	+0.7	158.75	162.17	+2.2
19	0.75	0	131.56	131.84	+0.2	183.89	184.29	+0.2
20		0.25	140.91	140.07	-0.6	186.53	187.76	+0.7
21		0.5	152.80	152.69	-0.1	189.39	191.38	+1.0
22		0.75	168.72	169.71	+0.6	192.25	194.99	+1.4
23		0.9	181.39	182.02	+0.3	194.12	197.16	+1.6
24		1	191.80	192.95	+0.6	195.31	198.75	+1.8
25	0.9	0	152.21	152.79	+0.4	224.14	225.43	+0.6
26		0.25	163.06	162.38	-0.4	228.10	230.70	+1.1
27		0.5	176.83	177.10	+0.2	232.43	236.19	+1.6
28		0.75	195.20	196.94	+0.9	236.85	241.68	+2.0
29		0.9	209.71	211.30	+0.8	239.85	244.97	+2.1
30		1	221.60	224.03	+1.1	241.88	247.39	+2.3
31	1	0	177.21	178.46	+0.7	274.29	278.31	+1.5
32		0.25	189.64	189.52	-0.1	280.58	287.05	+2.3
33		0.5	205.31	206.48	+0.6	287.26	296.15	+3.1
34		0.75	225.98	229.35	+1.5	294.76	305.25	+3.6
35		0.9	242.10	245.90	+1.6	299.71	310.71	+3.7
36		1	255.13	260.57	+2.1	303.01	314.71	+3.9

Table 9. Comparison of $M_{cr}(\kappa_\omega, \kappa_u)$

Item	Static scheme	κ_ω [-]	κ_u [-]	M_{cr} [kNm]		% [-]
				LTBeamN	Formula (5)	VI-V
I	II	III	IV	V	VI	VII
1		1	0	191.82	194.13	+1.2
2	IPE300, L = 5m, z_g = +h/2	0.9	0.25	176.14	176.28	+0.1
3		0.75	0.5	163.71	163.62	-0.1
4		0.5	0.75	153.56	154.22	+0.4
5	κ_v = 0	0.25	0.9	148.32	149.75	+1.0
6		0	1	145.13	148.42	+2.3
7		1	0	167.43	172.76	+3.2
8	IPE300, L = 5m, z_g = +h/2	0.9	0.25	144.20	148.57	+3.0
9		0.75	0.5	124.04	127.32	+2.6
10		0.5	0.75	106.36	109.06	+2.5
11	κ_v = 1	0.25	0.9	96.64	99.71	+3.2
12		0	1	90.56	94.37	+4.2
13		1	0	180.60	181.99	+0.8
14	IPE300, L = 5m, z_g = +h/2	0.9	0.25	165.91	165.75	-0.1
15		0.75	0.5	155.21	156.03	+0.5
16		0.5	0.75	147.73	149.90	+1.5
17	κ_v = 0	0.25	0.9	144.75	147.28	+1.8
18		0	1	143.50	147.40	+2.7
19		1	0	324.26	343.90	+6.1
20	IPE300, L = 5m, z_g = +h/2	0.9	0.25	269.26	283.42	+5.3
21		0.75	0.5	223.23	233.50	+4.6
22		0.5	0.75	184.73	192.91	+4.4
23	κ_v = 1	0.25	0.9	164.77	172.82	+4.9
24		0	1	152.46	161.39	+5.9

beam M_{max} occurs in the middle of the span, and for a bilaterally fixed beam on the support). The occurrence of such situations when changing the degree of fixity to bending M_y in the nodes is described in [9].

Figure 6 shows the variation surfaces of M_{cr} as a function of the indexes κ_ω and κ_u for an IPE300 beam with a span of L = 5 m, uniformly loaded at the height of the top (z_g = +h/2) and the bottom (z_g = -h/2) flange of the cross-section. Figure 6a shows the $M_{cr}(\kappa_\omega, \kappa_u)$ plots for a simply supported beam (κ_v = 0), and Figure 6b shows the $M_{cr}(\kappa_\omega, \kappa_u)$ plots for a bilaterally fixed beam (κ_v = 1). The critical moments were estimated using formula (5).

The analysis of the plots (Fig. 6) allows us to conclude that with the increase of the κ_ω and κ_u indexes, the value of the critical moment of lateral torsional buckling of the beam increases. An increase in the κ_ω index (from 0 to 1), for a constant value of the κ_u index, causes a strongly non-linear increase in M_{cr}. However, with the increase of the κ_u index (from 0 to 1), at a constant value of the κ_ω index, an essentially linear increase in M_{cr} was observed with the load applied to the top flange (z_g = +h/2) and a mildly non-linear increase in M_{cr} with the load suspended to the bottom flange (z_g = -h/2). Analogous observations of $M_{cr}(\kappa_\omega, \kappa_u)$ variation curves were found for all analyzed beams.

Table 10 presents the maximum percentage differences of $M_{cr}(\kappa_\omega, \kappa_u)$ values estimated using the approximation formula (5) with the values obtained from the *LTBeamN* (FEM) software. The analysis took into account beams made of IPE300, HEA300, HEB300 sections (for L = 5 and 7 m) and IPE500, HEA500, HEB500 sections (for L = 8 and 10 m).

For simply supported beams κ_v = 0 (rows 1 to 9), the percentage differences (col. IV) of M_{cr} estimates compared to FEM ranged from -3.3 (uniformly distributed load) to +4.0%

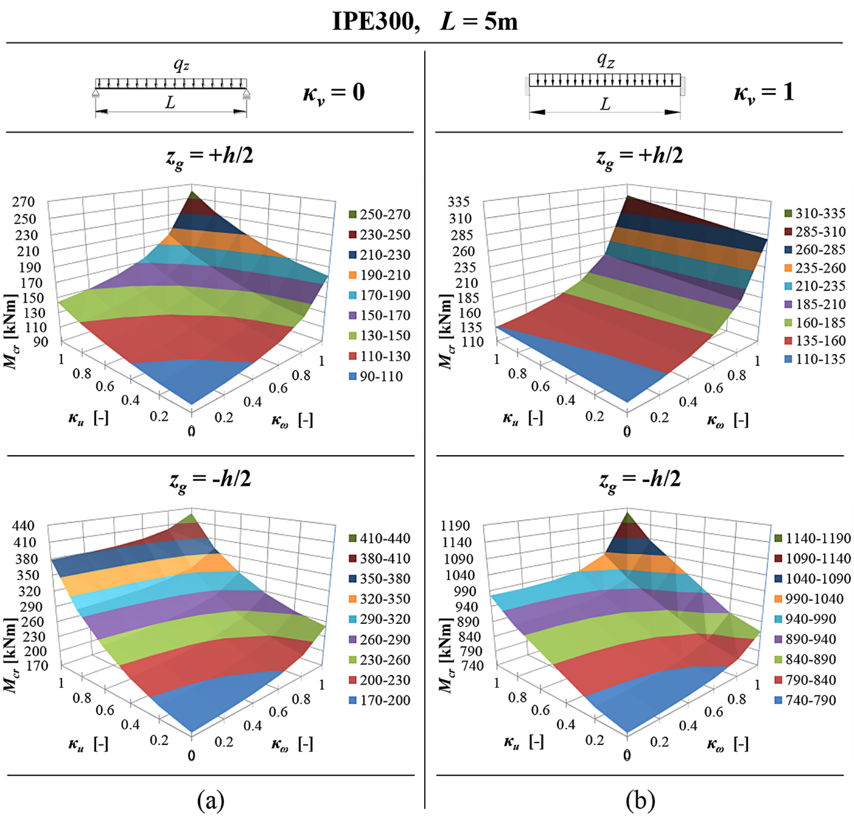
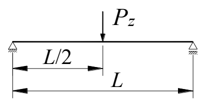
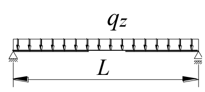
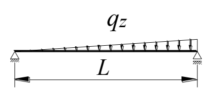
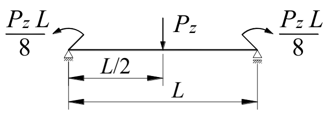
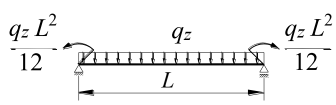
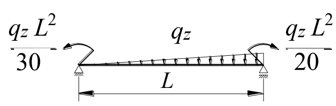


Figure 6. Trends in $M_{cr}(\kappa_\omega, \kappa_u)$ variation according to formula (5) for a uniformly loaded beam: (a) simply supported beam ($\kappa_v = 0$), (b) bilaterally fixed beam ($\kappa_v = 1$)

Table 10. Percentage differences in $M_{cr}(\kappa_\omega, \kappa_u)$ values for all analyzed beams

Item	Static scheme	Differences in M_{cr}	
		Formula (5) vs. *LTBeamN*	
		Ordinate of load application	% [-]
I	II	III	IV
1		$z_g = +h/2$	from -0.4 to +3.7%
2		$z_g = 0$	from -1.5 to +2.2%
3		$z_g = -h/2$	from -3.0 to +2.2%
4		$z_g = +h/2$	from -0.7 to +2.4%
5		$z_g = 0$	from -1.3 to +1.9%
6		$z_g = -h/2$	from -3.3 to +3.1%
7		$z_g = +h/2$	from -0.3 to +3.4%
8		$z_g = 0$	from -1.2 to +2.5%
9		$z_g = -h/2$	from -3.0 to +4.0%
10		$z_g = +h/2$	from +0.4 to +6.8%
11		$z_g = 0$	from -1.2 to +4.7%
12		$z_g = -h/2$	from -4.8 to +8.0%
13		$z_g = +h/2$	from -0.3 to +5.1%
14		$z_g = 0$	from -0.6 to +3.8%
15		$z_g = -h/2$	from -2.4 to +5.1%
16		$z_g = +h/2$	from +2.7 to +7.2%
17		$z_g = 0$	from +0.7 to +6.2%
18		$z_g = -h/2$	from -4.2 to +8.9%

(triangularly distributed load). In the case of bilaterally fixed beams $\kappa_v = 1$ (rows 10 to 18), differences were obtained (col. IV) in the range from -4.8 (concentrated force load) to +8.9% (triangular distributed load). Slightly higher differences in the estimation of the M_{cr} value of bilaterally fixed beams ($\kappa_v = 1$) according to formula (5) in comparison with FEM (*LTBeamN*) are related, among others, to the location of the maximum moment M_y of the beam (moment over the support), identified with the estimated critical moment [9], and additionally, in the case of triangular loading, to the relatively asymmetric form of stability loss.

Table 11 compares the percentage differences of the current estimate of $M_{cr}(\kappa_\omega, \kappa_u)$ from formula (5) compared to *LTBeamN* (FEM) with the percentage differences of the estimate of $M_{cr}(\kappa_\omega, \kappa_u)$ given in [12], formula (4) compared to *LTBeam* (FEM).

The analysis of the values given in Table 11 shows that the accuracy of the $M_{cr}(\kappa_\omega, \kappa_u)$ estimate from formula (5), in relation to FEM, is comparable to the estimate obtained in [12] according to formula (4). It should be noted that the currently proposed formula (5) has a simpler analytical form compared to formula (4) [12]. The mathematical homogeneity and the method of constructing approximation formulas are in this case analogous to the procedure proposed in [9], which may allow its extension to even more complicated cases of elastic restraint of the beam in the support nodes.

Table 12 compares the $M_{cr}(\kappa_\omega, \kappa_u)$ values for a beam with a span of $L = 7$ m, made of IPE300, HEA300 and HEB300 sections. The selected sections have a similar cross-sectional height (for HEA 300, $h = 290$ mm), corresponding to often encountered in engineering practice beams, e.g. in classical composite ceilings (steel – concrete). In such a case, the lateral torsional buckling of the steel beam may occur, for example, during the phase of placing the concrete mixture on the formwork of the ceiling slab. For a beam uniformly loaded at the height of the top flange ($z_g = +h/2$), two extreme fixing conditions at M_y were assumed: a) simply support (rows 1–9), b) full fixing (rows 10–18). The critical moments were determined using formula (5).

The obtained M_{cr} (Table 12) confirm the significant influence of the flexural-torsional characteristics of the beam cross-section (I_z, I_t, I_ω) on the elastic critical resistance. The lowest M_{cr} values were obtained for the IPE300 narrow flange beam (col. V). Changing the cross-section to the HEA300 profile with a wide flange (col. VI) resulted in an average six-fold (col. VII) increase in the critical moment. The highest M_{cr} was obtained for the HEB300 beam (col. VIII), which was on average +60% higher than that obtained for HEA300 (col. X).

In turn, Table 13 compares the $M_{cr}(\kappa_\omega, \kappa_u)$ values of the same beams (IPE300, HEA300, HEB300) with a span of $L = 7$ m when loaded with a concentrated force in the middle of the span for $z_g = +h/2$. The critical moments were determined from formula (5).

The obtained M_{cr} (Table 13) confirm (analogously to the results from Table 12) the significant influence of the flexural-torsional characteristics of the beam cross-section (I_z, I_t, I_ω) on the elastic critical resistance. The change from a narrow flange I-section to a wide flange H-section, i.e. from IPE300 to HEA300 and HEB300,

Table 11. Comparison of percentage differences in the $M_{cr}(\kappa_\omega, \kappa_u)$ values obtained in the current work with the percentage differences obtained in the work [12]

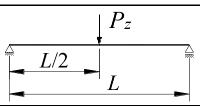
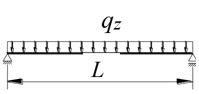
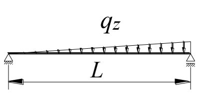
Item	Static scheme	Differences in M_{cr}	
		Formula (5) vs. *LTBeamN*	Formula (4) [12] vs. *LTBeam*
I	II	III	IV
1		from -3.0 to +3.7%	from -3.8 to +4.1%
2		from -3.3 to +3.1%	from -3.0 to +2.6%
3		from -3.0 to +4.0%	from -2.3 to +3.3%

Table 12. Comparison of $M_{cr}(\kappa_\omega, \kappa_u)$

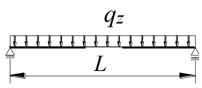
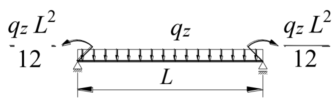
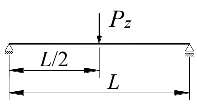
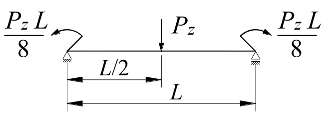
Item	Static scheme	κ_ω [-]	κ_u [-]	M_{cr} [kNm] acc. Formula (5)					
				IPE300	HEA300	% [-] VI-V	HEB300	% [-] VIII-V	% [-] VIII-VI
I	II	III	IV	V	VI	VII	VIII	IX	X
1	 L = 7m z_g = +h/2 κ_v = 0	0.25	0.25	74.67	532.26	+613	861.44	+1054	+61.8
2			0.5	82.05	577.92	+604	940.08	+1046	+62.7
3			0.75	92.00	639.46	+595	1046.07	+1037	+63.6
4		0.5	0.25	78.81	585.48	+643	926.81	+1076	+58.3
5			0.5	86.50	636.19	+635	1011.22	+1069	+58.9
6			0.75	96.87	704.55	+627	1125.00	+1061	+59.7
7		0.75	0.25	87.01	684.47	+687	1052.04	+1109	+53.7
8			0.5	95.41	744.92	+681	1148.21	+1103	+54.1
9			0.75	106.73	826.40	+674	1277.85	+1097	+54.6
10	 L = 7m z_g = +h/2 κ_v = 1	0.25	0.25	100.36	640.34	+538	1084.60	+981	+69.4
11			0.5	102.79	651.31	+534	1106.18	+976	+69.8
12			0.75	105.23	662.29	+529	1127.76	+972	+70.3
13		0.5	0.25	108.08	728.32	+574	1198.25	+1009	+64.5
14			0.5	110.58	740.20	+569	1220.87	+1004	+64.9
15			0.75	113.08	752.09	+565	1243.50	+1000	+65.3
16		0.75	0.25	124.06	902.11	+627	1427.39	+1051	+58.2
17			0.5	127.08	918.33	+623	1456.27	+1046	+58.6
18			0.75	130.10	934.54	+618	1485.16	+1042	+58.9

Table 13. Comparison of $M_{cr}(\kappa_\omega, \kappa_u)$

Item	Static scheme	κ_ω [-]	κ_u [-]	M_{cr} [kNm] acc. Formula (5)					
				IPE300	HEA300	% [-] VI-V	HEB300	% [-] VIII-V	% [-] VIII-VI
I	II	III	IV	V	VI	VII	VIII	IX	X
1	 L = 7m z_g = +h/2 κ_v = 0	0.25	0.25	84.24	592.85	+604	964.73	+1045	+62.7
2			0.5	90.29	628.05	+596	1027.01	+1037	+63.5
3			0.75	98.44	675.50	+586	1110.96	+1029	+64.5
4		0.5	0.25	88.40	648.66	+634	1032.03	+1067	+59.1
5			0.5	94.62	687.33	+626	1097.99	+1060	+59.7
6			0.75	103.00	739.45	+618	1186.91	+1052	+60.5
7		0.75	0.25	96.56	750.57	+677	1158.84	+1100	+54.4
8			0.5	103.23	796.09	+671	1232.76	+1094	+54.9
9			0.75	112.22	857.44	+664	1332.40	+1087	+55.4
10	 L = 7m z_g = +h/2 κ_v = 1	0.25	0.25	70.81	456.28	+544	769.88	+987	+68.7
11			0.5	71.98	459.77	+539	778.41	+981	+69.3
12			0.75	73.14	463.25	+533	786.94	+976	+69.9
13		0.5	0.25	74.94	508.50	+579	834.42	+1013	+64.1
14			0.5	76.05	511.83	+573	842.47	+1008	+64.6
15			0.75	77.15	515.16	+568	850.52	+1002	+65.1
16		0.75	0.25	83.22	606.66	+629	959.06	+1052	+58.1
17			0.5	84.45	611.12	+624	968.62	+1047	+58.5
18			0.75	85.67	615.59	+619	978.18	+1042	+58.9

resulted in a very significant percentage increase in the elastic critical resistance (Table 13).

Moreover, a comparison of the results from Tables 12 and 13 shows that for beams simply supported on M_y, the longitudinal distribution of the moment due to concentrated force load is more favourable than in the case of the longitudinal distribution of the moment due to uniform load (i.e. the M_{cr} value for concentrated force load is higher than for uniform load). In this case, M_{cr} occurs in the middle of the span, and the lateral torsional buckling is determined by the span zone of the beams. However, for beams fully fixed on M_y, the opposite phenomenon occurs, i.e. the value of the critical moment due to concentrated force load (M_{cr} in the middle of the beam span) is smaller than the absolute value of M_{cr} due to uniform load. In the latter case, M_{cr} occurs at the support and the lateral torsional buckling is determined by the zone near the support.

CONCLUSIONS

In this paper, a new, simpler approximation formula (5) (with the so-called integrated interaction coefficient) for the critical moment of lateral torsional buckling of beams elastically restrained in support nodes is proposed. The proposed solution allows to estimate the M_{cr} of the beam, taking into account: a) any degree of elastic restraint against warping (κ_ω) and lateral rotation (κ_u), b) free ($\kappa_v = 0$) or fixed ($\kappa_v = 1$) rotational restraint in the plane of the main bending M_y, c) load schemes often found in engineering practice, d) the ordinate of the transverse load application point (z_g) at the cross-section height.

The obtained solution allows for a relatively simple and sufficiently accurate, from a technical point of view, taking into account the actual behaviour of a steel beam sensitive to lateral torsional buckling, which is an element of a frame structure, e.g. a grillage or a frame. It can also be used to verify FEM numerical simulations.

The estimated values of $M_{cr}(\kappa_\omega,\ \kappa_u)$ allow to conclude that formula (5) gives a good approximation of the critical moments of lateral torsional buckling in comparison with FEM (e.g. *LT-BeamN*). The obtained percentage differences for the analyzed beams show that from the engineering point of view the proposed solution gives a sufficient approximation of the values determined by numerical simulations.

In practical calculations of the critical moment it should be assumed that the calculations are carried out for the range κ_u from 0.1 to 0.9, which covers the majority of technically important cases of elastic restraint of beams at support nodes. In the extreme ranges, i.e. from 0 to 0.1 and from 0.9 to 1, the M_{cr} values should be linearly interpolated.

The calculations confirmed that the $M_{cr}(\kappa_\omega,\kappa_u)$ value is significantly influenced by the flexural-torsional characteristics (I_z, I_t, I_ω) of the beam cross-section. Replacing a narrow flange I-section, e.g. IPE300, with a wide flange H-section, e.g. HEA300, can allow for a several times increase in the critical elastic resistance of the beam while maintaining the appropriate structural height, e.g. of the ceiling. The support conditions of the beam in bending with respect to the major axis of the cross-section influence the value and location of the maximum moment M_{max} identified with the engineering interpretation of M_{cr}.

A new formula for $M_{cr}(\kappa_\omega,\ \kappa_u)$ gives a good estimate of the elastic critical resistance and has a simpler analytical notation. An additional advantage of this approach is the possibility of taking into account the full restraint of the beam in its bending plane ($\kappa_v = 1$).

The obtained solution is a starting point and a reference point for deriving approximation formulas for M_{cr} taking into account the elastic interaction of three elastic restraints, technically important from the point of view of designing frame structures, i.e. the restraints against: a) warping, b) lateral rotation, c) rotation in the bending plane M_y.

REFERENCES

1. Seif M., Schafer B.W. Local buckling of structural steel shapes. Journal of Constructional Steel Research. 2010, 66(10), 1232–1247, doi:10.1016/j.jcsr.2010.03.015.
2. Gardner L., Fieber A., Macorini L. Formulae for calculating elastic local buckling stresses of full structural cross-sections. Structures. 2019, 17, 2–20, doi:10.1016/j.istruc.2019.01.012.
3. Giżejowski M., Barszcz A., Wiedro P. Refined energy method for the elastic flexural-torsional buckling of steel H-section beam-columns. Part I: Formulation and Solution. Archives of Civil Engineering. 2023, LXIX(1), 513–537, doi:10.24425/ace.2023.144186.

4. Czubacki R., Lewiński T. Application of the Rayleigh quotient method in the analysis of stability of straight elastic bars. Archives of Civil Engineering. 2024, LXX(4), 85–98, doi:10.24425/ace.2024.151881.
5. Zhang L., Zhang Q., Tong G., Zhu Q. Elastic local buckling and width-to-thickness limits of I-beams incorporating flange–web interactions. Buildings. 2024, 14(347), 1–20, doi:10.3390/buildings14020347.
6. EN 1993-1-1, Eurocode 3: Design of Steel Structures - Part 1-1: General Rules and Rules for Buildings. European Committee for Standardization: Brussels, Belgium, 2005.
7. EN 1993-1-4, Eurocode 3: Design of Steel Structures - Part 1-4: General Rules - Supplementary Rules for Stainless Steels. European Committee for Standardization: Brussels, Belgium, 2006.
8. Belaid T., Slimani A., Ammari F., Boukhalfa D., Adman R. Formulation of the critical lateral buckling moment of steel beams under asymmetric loadings. Thin-Walled Structures. 2023, 182(110163), 1–11, doi:10.1016/j.tws.2022.110163.
9. Piotrowski R., Szychowski A. The effect of steel beam elastic restraint on the critical moment of lateral torsional buckling. Materials. 2022, 15(1275), 1–26, doi:10.3390/ma15041275.
10. Zhang H.-Y., Ho G.W.M., Liu S.-W., Chen L., Chan S.-L. Advanced line-finite-element for lateral-torsional buckling of beams with torsion and warping restraints. Journal of Constructional Steel Research. 2025, 224(109103), 1–17, doi:10.1016/j.jcsr.2024.109103.
11. Piotrowski R., Szychowski A. Lateral-torsional buckling of beams elastically restrained against warping at supports. Archives of Civil Engineering. 2015, LXI(4), 155–174, doi:10.1515/ace-2015-0042.
12. Piotrowski R., Szychowski A. Lateral torsional buckling of steel beams elastically restrained at the support nodes. Applied Sciences. 2019, 9(9), 1–17, doi:10.3390/app9091944.
13. Timoshenko S.P., Gere J.M. Theory of elastic stability. 2nd ed. McGraw-Hill: New York, NY, USA, 1961.
14. Lindner L., Gietzelt R. Stabilisierung von Biegeträgern Mit I-Profil Durch Angeschweiβte Kopfplatten. Stahlbau. 1984, 3, 69–74.
15. Gotluru B.P., Schafer B.W., Pekoz T. Torsion in thin-walled cold-formed steel beams. Thin - Walled Structures. 2000, 37(2), 127–145, doi:10.1016/S0263-8231(00)00011-2.
16. Pi Y.-L., Trahair N.S. Distortion and warping at beam supports. Journal of Structural Engineering. 2000, 126(11), 1279–1287, doi:10.1061/(ASCE)0733-9445(2000)126:11(1279).
17. LTBeamN Software, Version 2.0.1, CTICM. Available Online: https://www.cticm.com/logiciel/ltbeamn/ (Accessed on 1 October 2023).
18. Giżejowski M., Barszcz A., Wiedro P. Refined energy method for the elastic flexural-torsional buckling of steel H-section beam-columns. Part II: Comparison and Verification for Elements LTU and LTR. Archives of Civil Engineering. 2023, LXIX(2), 265–289, doi:10.24425/ace.2023.145267.
19. Bijak R. The lateral buckling of simply supported unrestrained bisymmetric I-shape beams. Archives of Civil Engineering. 2015, LXI(4), 127–140, doi:10.1515/ace-2015-0040.
20. Barszcz A.M., Giżejowski M.A., Papangelis J.P. Investigations into the Flexural-Torsional Buckling Behavior of Steel Open-Section Beam-Columns. Buildings. 2023, 13(307), 1–17, doi:10.3390/buildings13020307.
21. Lebastard M., Couchaux M., Santana M., Bureau A., Hjiaj M. Elastic Lateral-Torsional Buckling of Beams with Warping Restraints at Supports. Journal of Constructional Steel Research. 2022, 197(107410), 1–20, doi:10.1016/j.jcsr.2022.107410.