

## Appendix A: Detailed Derivation of Element Stiffness and Mass Matrices Using Hamilton's Principle

“This appendix provides the intermediate steps omitted from Section ‘Element stiffness and mass matrices’ for the sake of brevity. The derivation follows Hamilton's variational principle, which states that the time integral of the Lagrangian (kinetic energy minus potential energy) is stationary:

$$\int_{t_1}^{t_2} \delta(T - U) dt = 0 \quad (\text{A1})$$

where  $T$  is the kinetic energy,  $U$  is the strain energy (potential energy), and  $\delta$  denotes the first variation.

### A.1 Strain energy variation

The strain energy of the beam element is given by:

$$U_e = \frac{1}{2} \int_{V_e} \sigma_x \varepsilon_x dV = \frac{1}{2} \int_0^{L_e} \int_A \sigma_x \varepsilon_x dA dx \quad (\text{A2})$$

Substituting the constitutive relation  $\sigma_x = E(z)\varepsilon_x$  and the strain expression  $\varepsilon_x = \varepsilon_x^0 + z\kappa_x^b$  yields:

$$U_e = \frac{1}{2} \int_0^{L_e} \int_A E(z) (\varepsilon_x^0 + z\kappa_x^b)^2 dA dx \quad (\text{A3})$$

Expanding the square:

$$U_e = \frac{1}{2} \int_0^{L_e} \left[ \varepsilon_x^0 \int_A E(z) dA \cdot \varepsilon_x^0 + 2\varepsilon_x^0 \int_A z E(z) dA \cdot \kappa_x^b + \kappa_x^b \int_A z^2 E(z) dA \cdot \kappa_x^b \right] dx \quad (\text{A4})$$

Introducing the sectional rigidities defined in Eq. (8):

$$A = \iint_A E(z) dA, B = \iint_A zE(z) dA, D = \iint_A z^2 E(z) dA \quad (\text{A5})$$

the strain energy becomes:

$$U_e = \frac{1}{2} \int_0^{L_e} [A(\varepsilon_x^0)^2 + 2B\varepsilon_x^0\kappa_x^b + D(\kappa_x^b)^2] dx \quad (\text{A6})$$

### A.2 Kinetic energy variation

The kinetic energy of the element, including both axial and flexural motions, is:

$$T = \frac{1}{2} \int_{V_e} \rho(z) \left[ \left( \dot{u} - z \frac{\partial \dot{w}_b}{\partial x} \right)^2 + (\dot{w}_b)^2 \right] dV \quad (\text{A7})$$

Expanding the square term:

$$T = \frac{1}{2} \int_0^{L_e} \int_A \rho(z) \left[ \dot{u}^2 - 2z\dot{u} \frac{\partial \dot{w}_b}{\partial x} + z^2 \left( \frac{\partial \dot{w}_b}{\partial x} \right)^2 + (\dot{w}_b)^2 \right] dA dx \quad (\text{A8})$$

Integrating through the thickness using the inertia terms defined in Eq. (10):

$$I_0 = \iint_A \rho(z) dA, I_1 = \iint_A z\rho(z) dA, I_2 = \iint_A z^2 \rho(z) dA \quad (\text{A9})$$

the kinetic energy simplifies to:

$$T = \frac{1}{2} \int_0^{L_e} \left[ I_0 \dot{u}^2 - 2I_1 \dot{u} \frac{\partial \dot{w}_b}{\partial x} + I_2 \left( \frac{\partial \dot{w}_b}{\partial x} \right)^2 + I_0 \dot{w}_b^2 \right] dx \quad (\text{A10})$$

### A.3 Finite element discretization

The axial displacement  $u(x, t)$  and transverse displacement  $w_b(x, t)$  are interpolated using the shape functions from Eq. (5):

$$u(x, t) = \mathbf{N}_u(x) \mathbf{u}_e(t), w_b(x, t) = \mathbf{N}_w(x) \mathbf{u}_e(t) \quad (\text{A11})$$

where  $\mathbf{N}_u = [N_1, N_2, 0, 0, 0, 0]$ ,  $\mathbf{N}_w = [0, 0, N_3, N_4, N_5, N_6]$ , and  $\mathbf{u}_e = \{u_1, u_2, w_1, \theta_1, w_2, \theta_2\}^T$ .

The membrane strain and bending curvature are expressed in terms of the nodal displacements:

$$\varepsilon_x^0 = \frac{\partial u}{\partial x} = \mathbf{B}_u \mathbf{u}_e, \kappa_x^b = -\frac{\partial^2 w_b}{\partial x^2} = \mathbf{B}_w \mathbf{u}_e \quad (\text{A12})$$

where  $\mathbf{B}_u = \frac{d\mathbf{N}_u}{dx}$  and  $\mathbf{B}_w = -\frac{d^2\mathbf{N}_w}{dx^2}$ .

### A.4 Element stiffness matrix

Substituting the discretized strain fields into the strain energy expression and taking the variation yields:

$$\delta U_e = \delta \mathbf{u}_e^T \left( \int_0^{L_e} [\mathbf{A}\mathbf{B}_u^T \mathbf{B}_u + B(\mathbf{B}_u^T \mathbf{B}_w + \mathbf{B}_w^T \mathbf{B}_u) + D\mathbf{B}_w^T \mathbf{B}_w] dx \right) \mathbf{u}_e \quad (\text{A13})$$

Thus, the element stiffness matrix is:

$$\mathbf{K}_e = \int_0^{L_e} [\mathbf{A}\mathbf{B}_u^T \mathbf{B}_u + B(\mathbf{B}_u^T \mathbf{B}_w + \mathbf{B}_w^T \mathbf{B}_u) + D\mathbf{B}_w^T \mathbf{B}_w] dx \quad (\text{A14})$$

Performing the integration over the element length  $L_e$  using the explicit shape functions in Eq. (5) leads to the stiffness matrix given in Eq. (11).

### A.5 Element mass matrix

Similarly, substituting the discretized velocity fields into the kinetic energy expression and taking the variation gives:

$$\delta T = \delta \dot{\mathbf{u}}_e^T \left( \int_0^{L_e} [I_0 \mathbf{N}_u^T \mathbf{N}_u - I_1 (\mathbf{N}_u^T \mathbf{N}_{w,x} + \mathbf{N}_{w,x}^T \mathbf{N}_u) + I_2 \mathbf{N}_{w,x}^T \mathbf{N}_{w,x} + I_0 \mathbf{N}_w^T \mathbf{N}_w] dx \right) \dot{\mathbf{u}}_e \quad (\text{A15})$$

where  $\mathbf{N}_{w,x} = \frac{\partial \mathbf{N}_w}{\partial x}$ . The element mass matrix is therefore:

$$\mathbf{M}_e = \int_0^{L_e} [I_0 \mathbf{N}_u^T \mathbf{N}_u - I_1 (\mathbf{N}_u^T \mathbf{N}_{w,x} + \mathbf{N}_{w,x}^T \mathbf{N}_u) + I_2 \mathbf{N}_{w,x}^T \mathbf{N}_{w,x} + I_0 \mathbf{N}_w^T \mathbf{N}_w] dx \quad (\text{A16})$$

Carrying out the integration using the shape functions from Eq. (5) yields the mass matrix presented in Eq. (12). Note that when the material distribution is symmetric about the neutral axis,  $I_1 = 0$  and the coupling terms vanish.

#### **A.6 Assembly and global equations**

The element matrices are assembled into global matrices using the standard finite element assembly procedure. Applying Hamilton's principle and including the work done by the external force from the moving vehicle leads to the global equations of motion in Eq. (16). The coupling with the vehicle equations then produces the complete coupled system in Eq. (17) [45].