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On the Bahadur Representation of Quantiles for a Sample from ρ*-Mixing Structure Population

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ABSTRACT

In this paper, we establish the strong consistency and the Bahadur representation of sample quantiles for ρ*-mixing random variables. Additionally, the asymptotic normality and the Berry-Esseen bound of sample quantiles for ρ*-mixing random variables are presented. Moreover, numerical simulation is presented to ilustrate obtained results.

Keywords: quantiles, Bahadur representation, ρ*-mixing random variables, asymptotic normality, Berry-Esseen bound, simulation

INTRODUCTION

Mathematical modeling of phenomena is a very valuable way of describing phenomena occuring in nature and being a result of human activity. In the study of random phenomena we use statistical and probabilistic tools, therefore it is important to be able to determine the distribution and parameters of the tested feature on the basis of a sample from the studied population. Quantiles are a useful tool in many fields of science, especially economics and finance, where one of the most popular risk measures, the VaR (Value at Risk) measure, is based on the definition of the quantile. If $\{Y_k, k \geq 1\}$ is a strictly stationary dependent process with marginal distribution function F, then $1-p$ level VaR is defined as

$$VaR_p = \inf\{x : F(x) \geq p\}$$

for positive p close to 0. The estimation of quantiles is a popular topic in modern statistics researches.

Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a distribution function F. The p-th quantile of F is defined as $Q_p = \inf\{x : F(x) \geq p\}$, where $0 < p < 1$. Let $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I[X_i \leq x]$, $x \in \mathbb{R}$, $n \geq 1$ be the empirical distribution function, $Q_{n,p} = \inf\{x : F_n(x) \geq p\}$ be the sample p-th quantile. We put $Q_{n,p} = X_{n,\lfloor np \rfloor+1}$, where $(X_{n,1}, X_{n,2}, ..., X_{n,n})$ is the ordered sample of $(X_1, ..., X_n)$ and $\lfloor x \rfloor$ denotes integer part of x.

Bahadur [1] first established an elegant representation for sample quantile by means of empirical distribution function based on independent and identically distributed samples.

Theorem 1.1. *[1] Let $0 < p < 1$ and $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with distribution function F. Assume that F has at least two* *derivatives at some neighborhood of* Q_p *and* $F'(Q_p) = f(Q_p) > 0$*. Then*

$$Q_{n,p} = Q_p - \frac{F_n(Q_p) - p}{f(Q_p)} + O(n^{-\frac{3}{4}} \log n)) \quad a.s. \tag{1}$$

In many statistical models the elements in the sample are not always independent. Thus the assumption of independence should be replaced by the assumption that there is some structure of dependence in the sample. Hence, many researchers are investigating the Bahadur representation for sample quantiles in dependent samples. In papers [2], [3], [4], [5] and [6] φ-mixing sequences were analyzed, in [7], [8] and [9] α-mixing sequences were investigated. In [10] NA sequences and in [11] NOD sequences were discussed.

The aim of this article is to check whether the results obtained in the previously mentioned papers are still true in the case of ρ^*-mixing sequence of random variables.

Definition 1.1. *A sequence of random variables* $\{X_n, n \geq 1\}$ *is called* ρ^**-mixing, if the mixing coefficient*

$$\rho^*(n) = \sup\{\rho(S,T) \colon S, T \subset \mathbb{N}, dist(S,T) \geq n\} \to 0$$

as $n \to \infty$*, where*

$$\rho(S,T) = \sup\left\{ \frac{|Cov(X,Y)|}{\sqrt{Var(X)Var(Y)}} \colon X \in L_2(\sigma(S)), Y \in L_2(\sigma(T)) \right\},$$

$dist(S,T) = \min_{i \in S, j \in T} |j - i|$ *and* $\sigma(S)$ *and* $\sigma(T)$ *are the* σ*-fields generated by* $\{X_i, i \in S\}$ *and* $\{X_j, j \in T\}$*, respectively.*

Example 1.1. *Let* $\{\epsilon_n\}$ *be a sequence of i.i.d. random variables with zero mean and finite variance. Define* $X_n = \sum_{k=0}^{m} a_k \epsilon_{n-k}$ *for some positive integer* m *and constants* a_k*,* $k = 0, 1, ..., m$*. Then* $\{X_n\}$ *is known as a moving average process with older* m*. It can be easily verified that* $\{X_n\}$ *is a* ρ^**-mixing process.*

Example 1.2. *Let* $\{X_n, n \geq 1\}$ *be a strictly stationary, finite-state, irreducible and aperiodic Markov chain. Then it is a* ρ^**-mixing process with* $\rho^*(k) = o(e^{-Ck})$ *for some* $C > 0$*.*

Remark 1.1. *Note that increasing functions defined on disjoint subset of a* ρ^**-mixing field* $\{X_k, k \in N^d\}$ *with mixing coefficients* $\rho^*(s)$ *are also* ρ^**-mixing with coefficients not greater that* $\rho^*(s)$*.*

Numerous authors established a number of limit results for ρ^*-mixing sequences of random variables. For example in [12] the central limit theorem was presented. In [13], [14] and [15] the moment inequalities were obtained and in [16] the complete convergance of weighted sums for ρ^*-mixing sequences of random variables was investigated.

The following properties of ρ^*-mixing structures presented as lemmas will be significant in our subsequent discussions.

Lemma 1.2. *[17] Let* $q \geq 2$ *and* $\{X_n, n \geq 1\}$ *be a sequence of* ρ^**-mixing random variables with* $EX_n = 0$ *and* $E|X_n|^q < \infty$ *for every* $n \geq 1$*. The for all* $n \geq 1$*,*

$$E \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k \right|^q \leq C_q \left\{ \sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2 \right)^{\frac{q}{2}} \right\},$$

where $C_p > 0$ *depends only on* q *and the* ρ^**-mixing coefficients.*

Lemma 1.3. *Let $\{X_n, n \leq 1\}$ be a ρ^*-mixing sequence of random variables with finite variances, p and q be two integers. Let $\eta_l = \sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_i$ for $1 \leq l \leq k$. Then*

$$\left| E\exp\Big(i\sum_{l=1}^{k} t_l\eta_l\Big) - \prod_{l=1}^{k} E\exp(it_l\eta_l)\right|$$

$$\leq 4\sum_{1\leq l<j\leq k} |t_l||t_j|\Big\{ - Cov(\eta_l,\eta_j) + 16\rho^*(q)(Var(\eta_l))^{\frac{1}{2}}(Var(\eta_j))^{\frac{1}{2}}\Big\}.$$

Remark 1.2. *Based on Zhang [19], above Lemma is the special case of Theorem 3.3.*

Lemma 1.4. *[6] Suppose that $\{\xi_n, i \geq 1\}$ and $\{\eta_n, i \geq 1\}$ are two sequences of random variables. Let $\{\beta_n, n \geq 1\}$ be a positive constant sequence with $\beta_n \to 0$, as $n \to \infty$. If $\sup_{-\infty<u<\infty} |F_{\xi_n}(u) - \Phi(u)| \leq C\beta_n$, then for any $\varepsilon > 0$,*

$$\sup_{-\infty<u<\infty} |F_{\xi_n+\eta_n}(u) - \Phi(u)| \leq C[\beta_n + \varepsilon + P(|\eta_n| > \varepsilon)].$$

MAIN RESULTS

Let us consider the Bahadur representation of sample quantiles when the sample is taken from a ρ^*-mixing structure population.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with a common distribution function F and quantile Q_p. Assume that F possesses a positive continuous density f in some neighborhood $\mathfrak{D}_p$ of Q_p such that*

$$0 < \sup\{f(x); x \in \mathfrak{D}_p\} < \infty. \tag{2}$$

Then for any $\delta > \frac{1}{4}$

$$P\Big(\sup_{x\in\mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| = O(n^{-\frac{3}{4}+\delta}), \quad n \to \infty\Big) = 1,$$

where $\mathfrak{I}_n = [Q_p - c_0 n^{-\frac{1}{2}+\delta}, Q_p + c_0 n^{-\frac{1}{2}+\delta}]$ for some $c_0 > 0$.

Proof. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences defined as follows

$$a_n = c_0 n^{-\frac{1}{2}+\delta} \quad \text{for some } \ c_0 > 0, \quad b_n = \lfloor n^{\frac{1}{4}} \rfloor + 1$$

and

$$G_n(x) = F_n(x) - F_n(Q_p) - F(x) + p.$$

Then, for each $n \in \mathbb{N}$ and any integer j we define

$$\eta_{j,n} = Q_p + ja_nb_n^{-1}, \quad \alpha_{j,n} = F(\eta_{j+1,n}) - F(\eta_{j,n}) \quad \text{and} \quad \mathfrak{J}_{j,n} = [\eta_{j,n}, \eta_{j+1,n}].$$

Note that F_n and F are nondecreasing functions. Hence we get for $x \in \mathfrak{J}_{j,n}$

$$G_n(x) \leq F_n(\eta_{j+1,n}) - F_n(Q_p) - F(\eta_{j,n}) + p \leq G_n(\eta_{j+1,n}) + \alpha_{j,n}$$

and

$$G_n(x) \geq F_n(\eta_{j,n}) - F_n(Q_p) - F(\eta_{j+1,n}) + p \geq G_n(\eta_{j,n}) - \alpha_{j,n}.$$

Therefore

$$\sup_{x\in\Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \leq \max_{-b_n\leq j\leq b_n} \{|G_n(\eta_{j,n})|\} + \max_{-b_n\leq j\leq b_n-1} \{\alpha_{j,n}\}.$$

By The Mean Value Theorem and (2) we get

$$\alpha_{j,n} = F(\eta_{j+1,n}) - F(\eta_{j,n}) \leq C(\eta_{j+1,n} - \eta_{j,n}) = Ca_n b_n^{-1} \leq Cn^{-\frac{3}{4}+\delta}.$$

Hence, we have

$$\sum_{n=1}^{\infty} P\Big(\sup_{x\in\Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{3}{4}+\delta}\Big)$$

$$\leq C\sum_{n=1}^{\infty} P\Big(\max_{-b_n\leq j\leq b_n} |G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{3}{4}+\delta}\Big).$$

Additionally, we have that

$$G_n(\eta_{j,n}) = F_n(\eta_{j,n}) - F_n(Q_p) - F(\eta_{j,n}) + p = \frac{1}{n}\sum_{i=1}^{n}\Big(Y_i^{Q_p} - Y_i^{(j,n)}\Big)$$

where $Y_i^{Q_p} = E(I[X_i \leq Q_p]) - I[X_i \leq Q_p]$ and $Y_i^{(j,n)} = E(I[X_i \leq \eta_{j,n}]) - I[X_i \leq \eta_{j,n}]$, $-b_n \leq j \leq b_n$ are ρ^*-mixing random variables.

From Markov's inequality and Lemma 1.2 that for $r > \max\{2, \frac{5}{4\delta-1}\}$

$$\sum_{n=1}^{\infty} P\Big(\sup_{x\in\Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{3}{4}+\delta}\Big)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} P\Big(|G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{3}{4}+\delta}\Big) = C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} P\Big(|\sum_{i=1}^{n}\Big(Y_i^{Q_p} - Y_i^{(j,n)}\Big)| \geq \frac{c_0}{2} n^{\frac{1}{4}+\delta}\Big)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} P\Big(|\sum_{i=1}^{n} Y_i^{Q_p}| + |\sum_{i=1}^{n} Y_i^{(j,n)}| \geq \frac{c_0}{2} n^{\frac{1}{4}+\delta}\Big)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} \Big(P\Big(|\sum_{i=1}^{n} Y_i^{Q_p}| \geq \frac{c_0}{4} n^{\frac{1}{4}+\delta}\Big) + P\Big(|\sum_{i=1}^{n} Y_i^{(j,n)}| \geq \frac{c_0}{4} n^{\frac{1}{4}+\delta}\Big)\Big)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} \Big[\frac{E\Big(|\sum_{i=1}^{n} Y_i^{Q_p}|\Big)^r}{(n^{\frac{1}{4}+\delta})^r} + \frac{E\Big(|\sum_{i=1}^{n} Y_i^{(j,n)}|\Big)^r}{(n^{\frac{1}{4}+\delta})^r}\Big]$$

$$\leq C\sum_{n=1}^{\infty} 2b_n n^{\frac{r}{4}-\delta r} \leq C\sum_{n=1}^{\infty} n^{\frac{1}{4}+\frac{r}{4}-\delta r} < \infty.$$

□

The next theorem presents the strong consistency of $Q_{n,p}$ i.e. of an estimator of the quantile Q_p.

Theorem 2.2. *Suppose that assumptions of Theorem 2.1 hold. We assume that $f'(x)$ is defined in some neighborhood $\mathfrak{D}_p$ of Q_p,*

$$f'(x) < M, \quad x \in \mathfrak{D}_p. \tag{3}$$

Then for any $0 < \delta < \frac{1}{2}$

$$P\Big(Q_{n,p} - Q_p = o(n^{-\frac{1}{2}+\delta}), \quad as \;\; n \to \infty\Big) = 1. \tag{4}$$

Proof. We note that

$$\sum_{n=1}^{\infty} P\Big[|Q_{n,p} - Q_p| \geq \varepsilon n^{-\frac{1}{2}+\delta}\Big]$$

$$= \sum_{n=1}^{\infty} P\Big[Q_{n,p} \geq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}\Big] + \sum_{n=1}^{\infty} P\Big[Q_{n,p} \leq Q_p - \varepsilon n^{-\frac{1}{2}+\delta}\Big] = I_1 + I_2.$$

Let $\xi_{ni} = I(X_i \leq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) - F(Q_p + \varepsilon n^{-\frac{1}{2}+\delta})$ for $1 \leq i \leq n$. Hence, we have

$$I_1 = \sum_{n=1}^{\infty} P\Big\{Q_{n,p} \geq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}\Big\} = \sum_{n=1}^{\infty} P\Big\{\sum_{i=1}^{n} I(X_i \leq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) < \lfloor np \rfloor + 1\Big\}$$

$$= \sum_{n=1}^{\infty} P\Big\{\sum_{i=1}^{n} \xi_{ni} < \lfloor np \rfloor + 1 - nF(Q_p + \varepsilon n^{-\frac{1}{2}+\delta})\Big\}. \tag{5}$$

Using Taylor's expansion: $F(Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) = p + f(Q_p)\varepsilon n^{-\frac{1}{2}+\delta} + o(n^{-\frac{1}{2}+\delta})$ we can obtain that there exists some constant $c(\varepsilon) > 0$, depending only on ε, such that for a sufficiently large n

$$\sum_{n=1}^{\infty} P\Big(\sum_{i=1}^{n} \xi_{ni} < \lfloor np \rfloor + 1 - nF(Q_p + \varepsilon n^{-\frac{1}{2}+\delta})\Big) \leq \sum_{n=1}^{\infty} P\Big(\sum_{i=1}^{n} \xi_{ni} < -c(\varepsilon) n^{\frac{1}{2}+\delta}\Big) \tag{6}$$

Hence, from (5),(6), Markov's inequality and Lemma 1.2, for $r > \max\{2, \frac{1}{\delta}\}$, we obtain that

$$I_1 \leq \sum_{n=1}^{\infty} P\Big(\sum_{i=1}^{n} \xi_{ni} < -c(\varepsilon) n^{\frac{1}{2}+\delta}\Big) \leq \sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} \xi_{ni}\Big| > c(\varepsilon) n^{\frac{1}{2}+\delta}\Big)$$

$$\leq C \sum_{n=1}^{\infty} n^{-(\frac{1}{2}+\delta)r} E\Big|\sum_{i=1}^{n} \xi_{ni}\Big|^r \leq C \sum_{n=1}^{\infty} n^{-(\frac{1}{2}+\delta)r} \Big[(nE\xi_{n1}^2)^{\frac{r}{2}} + nE|\xi_{n1}|^r\Big] \leq C \sum_{n=1}^{\infty} n^{-\delta r} < \infty.$$

It can be similarly shown that $I_2 = \sum_{n=1}^{\infty} P\Big[Q_{n,p} \leq Q_p - \varepsilon n^{-\frac{1}{2}+\delta}\Big] < \infty$.

By the Borel-Cantelli lemma we get thesis (4). □

Theorem 2.3. *Assume that assumptions of Theorem 2.2 hold. Then for any $0 < \delta \leq \frac{1}{4}$ we have,*

$$P\left(Q_{n,p} = Q_p - \frac{F_n(Q_p) - p}{f(Q_p)} + O(n^{-\frac{3}{4}+\delta}), \quad as \;\; n \to \infty\right) = 1. \tag{7}$$

Proof. We have that $F_n(Q_{n,p}) = n^{-1}(\lfloor np \rfloor + 1) \leq p + n^{-1}$. Using Taylor's expansion we obtain for $0 < \theta < 1$

$$F(Q_{n,p}) = p + f(Q_p)(Q_{n,p} - Q_p) + \frac{1}{2} f'(Q_p + \theta(Q_{n,p} - Q_p))(Q_{n,p} - Q_p)^2.$$

From (3) and Theorem 2.2 it follows that

$$|F_n(Q_{n,p}) - F(Q_{n,p}) + f(Q_p)(Q_{n,p} - Q_p)|$$
$$\leq \frac{1}{2}|f'(Q_p + \theta(Q_{n,p} - Q_p))|(Q_{n,p} - Q_p)^2 + n^{-1} = o(n^{-1+2\delta}). \tag{8}$$

By (8) and Theorem 2.1, we get that with probability 1,

$$|f(Q_p)(Q_{n,p} - Q_p) + F_n(Q_p) - p|$$
$$\leq |F_n(Q_{n,p}) - F(Q_{n,p}) + f(Q_p)(Q_{n,p} - Q_p)| + |F_n(Q_{n,p}) - F(Q_{n,p}) - (F_n(Q_p) - p)|$$
$$\leq o(n^{-1+2\delta}) + \sup_{x \in \Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| = O(n^{-\frac{3}{4}+\delta}),$$

which gives $f(Q_p)(Q_{n,p} - Q_p) + F_n(Q_p) - p = O(n^{-\frac{3}{4}+\delta})$, when $n \to \infty$. Then, we get (7). □

Now we focus on uniformly asymptotic normality of the sample quantile for ρ^*-mixing random variables. We will prove four lemmas which will be necessary in our further considerations. To this purpose, we will use the methods and notation previously used in [20] and [6]. Let $\{p_n, n \geq 1\}$ and $\{q_n, n \geq 1\}$ be sequences such that for $p_n \to \infty$, $q_n \to \infty$ as $n \to \infty$ and for sufficiently large n

$$p_n + q_n \leq n, \quad 0 < q_n p_n^{-1} \leq c < \infty. \tag{9}$$

Moreover, we assume

$$p_n^{-1} q_n \to 0, \qquad n^{-\frac{1}{2}} p_n^{\frac{1}{2}} \to 0, \qquad \sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n p_n^{-1} \to 0.$$

Put $\sigma_p^2 := Var[I(X_1 \leq Q_p)] + 2\sum_{j=1}^{\infty} Cov[I(X_1 \leq Q_p), I(X_j \leq Q_p)] > 0$ and

$$Y_{ni} = \frac{P(X_i \leq Q_p) - I(X_i \leq Q_p)}{\sqrt{n}\sigma_p}.$$

Note that Y_{ni} are also ρ^*- mixing and $E|Y_{ni}|^q \leq Cn^{-\frac{q}{2}}$.
Put $S_n := \sum_{i=1}^{n} Y_{ni} = \frac{\sqrt{n}(F(Q_p) - F_n(Q_p))}{\sigma_p}$.
Additionally let

$$y_{nm} = \sum_{i=m(p_n+q_n)-p_n-q_n+1}^{m(p_n+q_n)-q_n} Y_{ni}, \qquad y'_{nm} = \sum_{i=m(p_n+q_n)-q_n+1}^{m(p_n+q_n)} Y_{ni}.$$

Then $S_n = S'_n + S''_n = \sum_{m=1}^{k_n} y_{nm} + \sum_{m=1}^{k_n} y'_{nm}$, where $k_n = \Big\lfloor \frac{n}{p_n + q_n} \Big\rfloor + 1$.

Set

$$\gamma_{1n} := p_n^{-1} q_n, \qquad \gamma_{2n} := \sum_{j=1}^{n-1} \frac{j}{n} \rho^*(j) + \sum_{j=n}^{\infty} \rho^*(j),$$

$$\gamma_{3n} := \sum_{t=q_n}^{\infty} \rho^*(t), \qquad \gamma_{4n} := n^{-\frac{1}{2}} p_n^{\frac{1}{2}}, \qquad \gamma_{5n} := \Big(\sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n p_n^{-1} \Big)^{\frac{1}{3}}.$$

Lemma 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with a common distribution function F and a density function f continuous in some neighborhood $\mathfrak{D}_p$ of Q_p satisfying (2)-(3). Let $\{p_n, n \geq 1\}$ and $\{q_n, n \geq 1\}$ satisfy (9). Then for any $r \geq 2$,*

$$E|S''_n|^r \leq C(\gamma_{1n})^{\frac{r}{2}} \tag{10}$$

and

$$P\Big(|S''_n| > (\gamma_{1n})^{\frac{r}{2(1+r)}}\Big) \leq C(\gamma_{1n})^{\frac{r}{2(1+r)}}. \tag{11}$$

Proof. From Lemma 1.2 we get

$$E|S''_n|^r = E\Big| \sum_{m=1}^{k_n} \sum_{i=m(p_n+q_n)-q_n+1}^{m(p_n+q_n)} Y_{ni} \Big|^r$$

$$\leq C\Big\{ \Big[\sum_{m=1}^{k_n} \sum_{i=m(p_n+q_n)-q_n+1}^{m(p_n+q_n)} EY_{ni}^2 \Big]^{\frac{r}{2}} + \sum_{m=1}^{k_n} \sum_{i=m(p_n+q_n)-q_n+1}^{m(p_n+q_n)} E|Y_{ni}|^r \Big\}$$

$$\leq C\Big\{ \Big[(k_n q_n) EY_{n1}^2 \Big]^{\frac{r}{2}} + k_n q_n E|Y_{n1}|^r \Big\} \leq C\Big[n p_n^{-1} q_n n^{-1} \Big]^{\frac{r}{2}} = C(\gamma_{1n})^{\frac{r}{2}},$$

which proves (10). Using Markov inequality and (10) we get (11):

$$P\Big(|S''_n| > (\gamma_{1n})^{\frac{r}{2(1+r)}}\Big) \leq (\gamma_{1n})^{-\frac{r^2}{2(1+r)}} E|S''_n|^r \leq C(\gamma_{1n})^{\frac{r}{2(1+r)}}.$$

□

Lemma 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with a common continous distribution function F. Then*

$$|ES_n^2 - 1| = O(\gamma_{2n}).$$

Proof. By definition of S_n we have $|ES_n^2 - 1| = \Big| \frac{E\{\sqrt{n}[F(Q_p) - F_n(Q_p)]\}^2}{\sigma_p^2} - 1 \Big|$.

It suffices to prove that $|E\{\sqrt{n}[F(Q_p) - F_n(Q_p)]\}^2 - \sigma_p^2| = O(\gamma_{2n})$. Put $Z_i = I(X_i \leq Q_p)$. Then we obtain

$$E\Big\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [I(X_i \leq Q_p) - EI(X_i \leq Q_p)] \Big\}^2 = E\Big\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [Z_i - EZ_i] \Big\}^2$$

$$= \frac{1}{n}\Big[\sum_{i=1}^{n} VarZ_i + 2\sum_{j=1}^{n}\sum_{j+1}^{n} Cov(Z_i, Z_j)\Big] = VarZ_1 + 2\sum_{j=1}^{n}\Big(1-\frac{j}{n}\Big)Cov(Z_1, Z_j).$$

By the definitions of $\rho^*(i)$ and Z_i we have

$$|Cov(Z_1, Z_j)| = \frac{|Cov(Z_1, Z_j)|}{(VarZ_1)^{1/2}(VarZ_j)^{1/2}} \cdot (VarZ_1)^{1/2}(VarZ_j)^{1/2}$$

$$\leq \rho^*(j) \cdot (VarZ_1)^{1/2}(VarZ_j)^{1/2} \leq C\rho^*(j). \tag{12}$$

Therefore we can state

$$|E\{\sqrt{n}[F(Q_p) - F_n(Q_p)]\}^2 - \sigma_p^2|$$

$$= \Big|VarZ_1 + 2\sum_{j=1}^{n}\Big(1-\frac{j}{n}\Big)Cov(Z_1, Z_j) - VarZ_1 - 2\sum_{j=1}^{\infty} Cov(Z_1, Z_j)\Big|$$

$$\leq 2\sum_{j=1}^{n}\frac{j}{n}|Cov(Z_1, Z_j)| + 2\sum_{j=n+1}^{\infty}|Cov(Z_1, Z_j)| \leq C\Big(\sum_{j=1}^{n-1}\frac{j}{n}\rho^*(j) + \sum_{j=n}^{\infty}\rho^*(j)\Big) = C(\gamma_{2n}).$$

□

Put

$$B_n = \sum_{i=1}^{k_n} Var(y_{ni}). \tag{13}$$

Lemma 2.6. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with a common continuous distribution function F and mixing coefficients $\{\rho^*(n), n \geq 1\}$satisfying $\sum_{n=1}^{\infty}\rho(n) < \infty$. Then*

$$|B_n - 1| = O\Big(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n}\Big).$$

Proof. We will use the following properties.

$$E(S_n')^2 = E(\sum_{i=1}^{k_n} y_{ni})^2 = \sum_{i=1}^{k_n} E(y_{ni})^2 + 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n} Cov(y_{ni}, y_{nj})$$

$$= B_n + 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n} Cov(y_{ni}, y_{nj}). \tag{14}$$

Hence, from (14) we have

$$B_n = E(S_n')^2 - 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n} Cov(y_{ni}, y_{nj}). \tag{15}$$

By (15) we obtain

$$|B_n - 1| = |E(S_n')^2 - 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n} Cov(y_{ni}, y_{nj}) - 1|$$

$$\leq |E(S_n')^2 - 1| + 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n}|Cov(y_{ni}, y_{nj})| = I_1 + I_2. \tag{16}$$

Using Lemma 2.5 we have

$$E(S_n')^2 = E(S_n - S_n'')^2 = ES_n^2 - 2E(S_nS_n'') + E(S_n'')^2$$

$$= E(S_n'')^2 - 2E(S_nS_n'') + 1 + O(\gamma_{2n}). \tag{17}$$

Additionally, by (17), Hölder's inequality and Lemma 2.4 we get

$$I_1 = |E(S_n')^2 - 1| \leq E(S_n'')^2 + 2(ES_n^2)^{\frac{1}{2}}(E(S_n'')^2)^{\frac{1}{2}} + O(\gamma_{2n}) = O(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n}). \tag{18}$$

Based on (12) we get

$$I_2 = 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n}|Cov(y_{ni}, y_{nj})|$$

$$\leq 2\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n}\sum_{s=i(p_n+q_n)-p_n-q_n+1}^{i(p_n+q_n)-q_n}\sum_{t=j(p_n+q_n)-p_n-q_n+1}^{j(p_n+q_n)-q_n}|Cov(Y_{ns}, Y_{nt})|.$$

$$\leq Cn^{-1}\sum_{i=1}^{k_n-1}\sum_{j=i+1}^{k_n}\sum_{s=i(p_n+q_n)-p_n-q_n+1}^{i(p_n+q_n)-q_n}\sum_{t=j(p_n+q_n)-p_n-q_n+1}^{j(p_n+q_n)-q_n}\rho(t-s)$$

$$\leq Cn^{-1}\sum_{i=1}^{k_n-1}\sum_{s=i(p_n+q_n)-p_n-q_n+1}^{i(p_n+q_n)-q_n}\sum_{t=q_n}^{\infty}\rho(t) \leq Cn^{-1}k_np_n\sum_{t=q_n}^{\infty}\rho(t) = C\gamma_{3n}. \tag{19}$$

By (16), (18) and (19) $|B_n - 1| = O(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n})$. □

Remark 2.1. *From Lemma 2.6 it follows $B_n \leq C$.*

Assume that $\{y_{nm}^*, 1 \leq m \leq k_n\}$ are independent copies of $\{y_{nm}, 1 \leq m \leq k_n\}$.

Put $S_n^* := \sum_{m=1}^{k_n} y_{nm}^*$. We see that $B_n^* = \sum_{m=1}^{k_n} Var(y_{nm}^*) = \sum_{m=1}^{k_n} Var(y_{nm}) = B_n$, and $F_{S_n^*}(u) = F_{\frac{S_n^*}{\sqrt{B_n^*}}}\Big(\frac{u}{\sqrt{B_n}}\Big)$.

Lemma 2.7. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with a common continuous distribution function F. Then*

$$\sup_{-\infty<u<\infty}\Big|F_{\frac{S_n^*}{\sqrt{B_n^*}}}(u) - \Phi(u)\Big| = O(\gamma_{4n}) \tag{20}$$

and

$$\sup_{-\infty<u<\infty}\Big|F_{S_n'}(u) - F_{S_n^*}(u)\Big| = O(\gamma_{4n} + \gamma_{5n}). \tag{21}$$

Proof. By Berry-Esseen theorem we get

$$\sup_{-\infty<u<\infty}\left|F_{\frac{T_n}{\sqrt{B_n}}}(u)-\Phi(u)\right|\le CB_n^{-\frac{3}{2}}\sum_{m=1}^{k_n}E|y^*_{nm}|^3.$$

Hence, by Remark 2.1 it is sufficient to show that $\sum_{m=1}^{k_n}E|y^*_{nm}|^3=O(\gamma_{4n})$.

From Lemma 1.2 we get

$$\sum_{m=1}^{k_n}E|y^*_{nm}|^3=\sum_{m=1}^{k_n}E|y_{nm}|^3=\sum_{m=1}^{k_n}E\left|\sum_{i=m(p_n+q_n)-p_n-q_n+1}^{m(p_n+q_n)-q_n}Y_{ni}\right|^3$$

$$\le Ck_n\Big[p_nE|Y_{n1}|^3+(p_nEY_{n1}^2)^{\frac{3}{2}}\Big]\le C\frac{n}{p_n}n^{-\frac{3}{2}}p_n^{\frac{3}{2}}=Cn^{-\frac{1}{2}}p_n^{\frac{1}{2}}=O(\gamma_{4n}). \tag{22}$$

The proof of (20) is completed.

Next, we will use the Esseen inequality (presented in [21])

$$\sup_{-\infty<u<\infty}|F_{S'_n}(u)-F_{S^*_n}(u)|$$

$$\le\int_{-T}^{T}\left|\frac{\chi(t)-\psi(t)}{t}\right|dt+T\sup_{-\infty<u<\infty}\int_{-\frac{c}{T}}^{\frac{c}{T}}|F_{S^*_n}(u+y)-F_{S^*_n}(u)|dy=A_1+A_2, \tag{23}$$

where $\chi(t)=E\exp(itS'_n)$, $\psi(t)=E\exp(itS^*_n)$ and $T,c>0$.

Note that $\psi(t)=\prod_{m=1}^{k_n}E\exp(ity^*_{nm})=\prod_{m=1}^{k_n}E\exp(ity_{nm})$. By Lemma 1.3, we obtain

$$|\chi(t)-\psi(t)|\le 8t^2\sum_{1\le m<j\le k_n}\Big\{-\operatorname{Cov}(y_{nm},y_{nj})+16\rho^*(q)(\operatorname{Var}(y_{nm}))^{\frac{1}{2}}(\operatorname{Var}(y_{nj}))^{\frac{1}{2}}\Big\}$$

$$\le 8t^2\Big\{\sum_{1\le m<j\le k_n}-\operatorname{Cov}(y_{nm},y_{nj})+\sum_{1\le m<j\le k_n}16\rho^*(q)(\operatorname{Var}(y_{nm}))^{\frac{1}{2}}(\operatorname{Var}(y_{nj}))^{\frac{1}{2}}\Big\}=8t^2\{I_1+I_2\}. \tag{24}$$

Then by (19) we get

$$I_1\le\sum_{1\le m<j\le k_n}|Cov(y_{nm},y_{nj})|\le C\sum_{t=q}^{\infty}\rho^*(t). \tag{25}$$

Additionally, using Lemma 1.2 we can show that

$$I_2\le C\rho^*(q_n)\sum_{1\le m<j\le k_n}(Ey_{nm}^2)^{\frac{1}{2}}(Ey_{nj}^2)^{\frac{1}{2}}\le C\rho^*(q_n)k_n^2[p_nEY_{1n}^2+p_nEY_{1n}^2]$$

$$\le C\rho^*(q_n)k_n^2p_nn^{-1}\le C\rho^*(q_n)np_n^{-1}. \tag{26}$$

By (24), (25), (26) we can easily show that

$$A_1 = \int_{-T}^{T} \left| \frac{\chi(t) - \psi(t)}{t} \right| dt \le CT^2 \Big(\sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n p_n^{-1} \Big). \tag{27}$$

Moreover, by (20) and Mean Value Theorem we have

$$\sup_u |F_{S_n^*}(u+y) - F_{S_n^*}(u)| \le \sup_u \Big| F_{\frac{S_n^*}{\sqrt{B_n^*}}} \Big(\frac{u+y}{\sqrt{B_n}} \Big) - \Phi\Big(\frac{u+y}{\sqrt{B_n}} \Big) \Big|$$

$$+ \sup_u \Big| \Phi\Big(\frac{u+y}{\sqrt{B_n}} \Big) - \Phi\Big(\frac{u}{\sqrt{B_n}} \Big) \Big| + \sup_u \Big| F_{\frac{T_n}{\sqrt{B_n^*}}} \Big(\frac{u}{\sqrt{B_n}} \Big) - \Phi\Big(\frac{u}{\sqrt{B_n}} \Big) \Big| \le C\Big(n^{-\frac{1}{2}} p_n^{\frac{1}{2}} + |y| \Big). \tag{28}$$

By (28) we get immediately that

$$A_2 = T \sup_{-\infty<u<\infty} \int_{-\frac{c}{T}}^{\frac{c}{T}} |F_{S_n^*}(u+y) - F_{S_n^*}(u)| dy \le C\Big(n^{-\frac{1}{2}} p_n^{\frac{1}{2}} + \frac{1}{T} \Big). \tag{29}$$

Hence, taking (27) and (29) we get

$$\sup_u |F_{S_n'(u)} - F_{S_n^*}(u)| \le C\Big(T^2 \Big(\sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n p_n^{-1} \Big) + n^{-\frac{1}{2}} p_n^{\frac{1}{2}} + \frac{1}{T} \Big). \tag{30}$$

Putting $T = \Big(\sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n p_n^{-1} \Big)^{-\frac{1}{3}}$ in (30) we get (21). □

Theorem 2.8. *Let $\{X_n, n \ge 1\}$ be a sequence of ρ^*-mixing random variables with a common continuous distribution function F and mixing coefficients $\{\rho^*(n), n \ge 1\}$ satisfying $\sum_{n=1}^{\infty} \rho(n) < \infty$. Suppose that assumptions (2)-(3) hold. Let sequences $\{p_n, n \ge 1\}$ and $\{q_n, n \ge 1\}$ satisfy (9). Then for any $r \ge 2$,*

$$\sup_{-\infty<u<\infty} \Big| P\Big(\frac{\sqrt{n}(Q_{n,p} - Q_p)}{\frac{\sigma_p}{f(Q_p)}} \le u \Big) - \Phi(u) \Big| = O\Big((\gamma_{1n})^{\frac{r}{2(1+r)}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n} \Big).$$

Proof. From Theorem 2.3 we have

$$Q_{n,p} - Q_p = \frac{F(Q_p) - F_n(Q_p)}{f(Q_p)} \quad \text{as} \quad n \to \infty.$$

Hence, it is enough to show that

$$\sup_u |F_{S_n}(u) - \Phi(u)| = O\Big((\gamma_{1n})^{\frac{r}{2(1+r)}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n} \Big).$$

It follows from Lemma 1.4, for $\varepsilon = (\gamma_{1n})^{\frac{r}{2(1+r)}}$, that

$$\sup_u |F_{S_n}(u) - \Phi(u)| \le \sup_u |F_{S_n'+S_n''}(u) - \Phi(u)| \le C[\beta_n + (\gamma_{1n})^{\frac{r}{2(1+r)}} + P(|S_n''| > (\gamma_{1n})^{\frac{r}{2(1+r)}})], \tag{31}$$

where based on Lemma 1.4 $\beta_n \to 0$ and $\sup_u |F_{S'_n}(u) - \Phi(u)| \leq C\beta_n$. Using Mean Value Theorem, we can obtain the form of β_n.

$$\sup_u |F_{S'_n}(u) - \Phi(u)| \leq \sup_u (|F_{S'_n}(u) - F_{S^*_n}(u)| + |F_{S^*_n}(u) - \Phi\Big(\frac{u}{\sqrt{B_n}}\Big)| + |\Phi\Big(\frac{u}{\sqrt{B_n}}\Big) - \Phi(u)|)$$

$$\leq \sup_u |F_{S'_n}(u) - F_{S^*_n}(u)| + \sup_u |F_{\frac{S^*_n}{\sqrt{B_n}}}\Big(\frac{u}{\sqrt{B_n}}\Big) - \Phi\Big(\frac{u}{\sqrt{B_n}}\Big)| + C\sup_u \Big|\frac{u}{\sqrt{B_n}}\Big| e^{-\frac{[u+\theta(\frac{u}{\sqrt{B_n}}-u)]^2}{2}} |\sqrt{B_n} - 1|.$$

By properties of function $f(x) = |x|e^{-x^2}$, one can see that

$$\sup_u \Big|\frac{u}{\sqrt{B_n}}\Big| e^{-\frac{[u+\theta(\frac{u}{\sqrt{B_n}}-u)]^2}{2}} \leq C.$$

Additionally, by Lemma 2.6 and Lemma 2.7 we get

$$\sup_u |F_{S'_n}(u) - \Phi(u)| \leq C|B_n - 1| + \sup_u |F_{S'_n}(u) - F_{S^*_n}(u)| + \sup_u |F_{\frac{S^*_n}{\sqrt{B_n}}}(u) - \Phi(u)|$$

$$= C(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n}).$$

Hence

$$\beta_n = C(\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n}). \tag{32}$$

. By assumptions of Theorem 2.8 $\gamma_{in} \to 0$, as $n \to \infty$ for $i = 1, 2, 3, 4, 5$.

Therefore relations (31), (32) and (11) imply

$$\sup_u |F_{S_n}(u) - \Phi(u)| \leq \gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n} + (\gamma_{1n})^{\frac{r}{2(1+r)}} + P\Big(|S''_n| > (\gamma_{1n})^{\frac{r}{2(1+r)}}\Big)$$

$$\leq C\Big[\gamma_{1n}^{\frac{1}{2}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n} + (\gamma_{1n})^{\frac{r}{2(1+r)}}\Big] \leq C\Big[(\gamma_{1n})^{\frac{r}{2(1+r)}} + \gamma_{2n} + \gamma_{3n} + \gamma_{4n} + \gamma_{5n}\Big].$$

□

Remark 2.2. *From Theorem 2.8 we get* $\dfrac{\sqrt{n}(Q_{n,p} - Q_p)}{\frac{\sigma_p}{f(Q_p)}} \xrightarrow{d} N(0,1)$ *as* $n \to \infty$.

Additionally, we can also obtain the following conclusions concerning the rate of normal approximation for different type of mixing coefficients.

Corollary 2.8.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of* ρ^**-mixing random variables with* $\rho^*(n) = O(n^{-\alpha})$, $\alpha > 1$ *and distribution function* F. *Suppose that assumptions (2) and (3) hold. Then for any* $0 < \kappa < \frac{1}{6}$,

$$\sup_{-\infty<u<\infty} \left| P\Big(\frac{\sqrt{n}(Q_{n,p} - Q_p)}{\frac{\sigma_p}{f(Q_p)}} \leq u\Big) - \Phi(u) \right| = O(n^{-\frac{1}{6}+\kappa}).$$

Proof. Let $p_n = \lfloor n^{\frac{2}{3}} \rfloor$, $q_n = \lfloor n^{\frac{1}{3}} \rfloor$, $\alpha > 1$. Let us note that for sufficiently large $r \geq 2$ we get

$$(\gamma_{1n})^{\frac{r}{2(1+r)}} \leq C[n^{-\frac{1}{3}}]^{\frac{r}{2(r+1)}} = O(n^{-\frac{1}{6}+\kappa}).$$

Moreover,

$$\gamma_{2n} \leq C\Big(\frac{1}{n}\sum_{j=1}^{n} j^{1-\alpha} + \sum_{j=n}^{\infty} j^{-\alpha}\Big) \leq Cn^{1-\alpha} = O(n^{-\frac{1}{6}+\kappa}),$$

$$\gamma_{3n} = \sum_{t=q_n}^{\infty} \rho(t) \leq Cn^{\frac{1-\alpha}{3}} = O(n^{-\frac{1}{6}+\kappa}), \qquad \gamma_{4n} = n^{-\frac{1}{2}} p_n^{\frac{1}{2}} \leq Cn^{-\frac{1}{6}} = O(n^{-\frac{1}{6}+\kappa}).$$

$$\gamma_{5n} = \Big(\sum_{t=q}^{\infty} \rho^*(t) + \rho^*(q_n) n^{\frac{1}{2}} p_n^{-\frac{1}{2}}\Big)^{\frac{1}{3}} \leq Cn^{\frac{1-\alpha}{9}} = O(n^{-\frac{1}{6}+\kappa}).$$

□

Corollary 2.8.2. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $\rho^*(n) = O(e^{-sn})$, for some $s > \frac{1}{2}$, a distribution function F and a density function f. Suppose that assumptions (2) and (3) Then for any $0 < \tau < \frac{1}{4}$*

$$\sup_{-\infty<u<\infty} \left| P\Big(\frac{\sqrt{n}(Q_{n,p} - Q_p)}{\frac{\sigma_p}{f(Q_p)}} \leq u\Big) - \Phi(u) \right| = O(n^{-\frac{1}{4}+\tau}).$$

Proof. Putting $p_n = \lfloor n^{\frac{1}{2}} \rfloor$, $q_n = \lfloor \log n \rfloor$ and using the standard estimations for any $0 < \delta < \frac{1}{2}$ we obtain that there exists $0 < \tau < \frac{1}{4}$ such that $\gamma_{kn} = O(n^{-\frac{1}{4}+\tau})$ as $k \in \{2, 3, 4, 5\}$ and $(\gamma_{1n})^{\frac{r}{2(1+r)}} = O(n^{-\frac{1}{4}+\tau})$.

□

SIMULATION

As already mentioned in the Example 1.1, the moving average process shows a ρ^*-mixing property. Let $\eta_i \overset{i.i.d.}{\sim} U\Big(-\sqrt{\frac{3}{m+1}}, \sqrt{\frac{3}{m+1}}\Big)$, where m is fixed positive integer. In this simulation we put $m = 10$. Then for each $i \geq 1$ $X_i = \sum_{k=0}^{m} \eta_{i+k}$ is a sequence of ρ^*-mixing random variables. Using R software we compute 1000 times statistic $U_n = \sqrt{n}(Q_{n,p} - Q_p)$. According to Corollary 2.2 statistic $U_n \xrightarrow{d} N\Big(0, \Big(\frac{\sigma_p}{f(Q_p)}\Big)^2\Big)$. To verify this we present Quantile-Quantile plots in Figures 1-4. for different sizes of samples respectively for $n = 200, 500, 1\,000, 2\,000$.

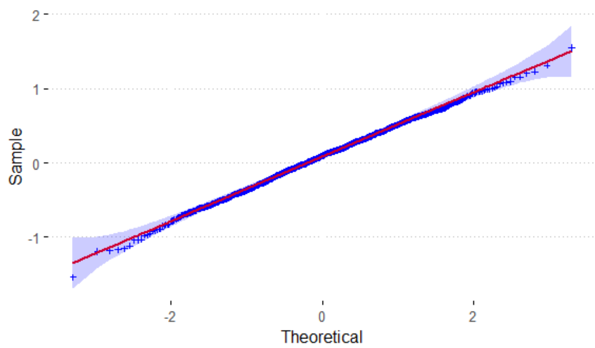

Figure 1: Sample $n = 200$

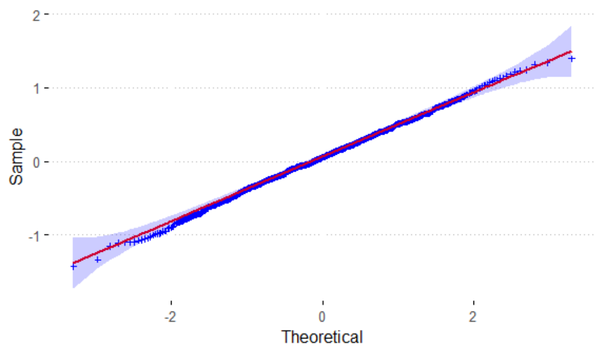

Figure 2: Sample $n = 500$

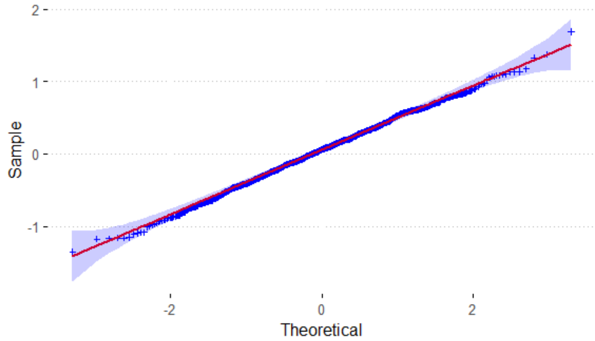

Figure 3: Sample $n = 1000$

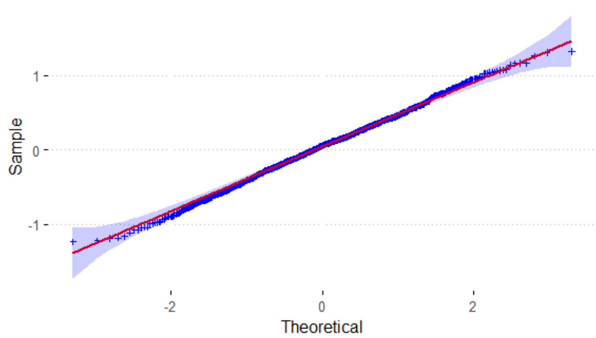

Figure 4: Sample $n = 2000$

To ilustrate result in Theorem 2.2 i.e. the consistency of the sample quantile we compute in every case Mean Squared Error (MSE) and bias for values of $Q_{n,p} - Q_p$.

Table 1: Bias and MSE of sample quantiles

	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
Bias	0.00562	0.00236	0.00147	0.00082
MSE	0.00097	0.00041	0.00021	0.00010

The table 1 shows that the bias and the MSE decreases as the sample size increases. This simulation basically agree with the main results established in section 2.

CONCLUSIONS

In our article, we obtained the Bahadur representation for sample quantiles from a population with ρ^*-mixing structure, thus we extend the scope of applicability to another population with a next dependent structure. We showed not only the consistency, the asymptotic normality and the Berry-Essen bound results about sample quantiles but also we provide the rate of convergence of sample quantiles to population counterparts. It was proved that the rate of normal approximation is $O(n^{-\frac{1}{6}+\kappa})$ for any $0 < \kappa < \frac{1}{6}$ if mixing coefficients satisfy $\rho(n) = O(n^{-\alpha})$ for some $\alpha > 1$ and $O(n^{-\frac{1}{4}+\tau})$ for any $0 < \tau < \frac{1}{4}$ if mixing coefficients decay exponentially. The presented simulation corresponds to the proven theorems. The simulation shows that the distribution of $Q_{n,p} - Q_p$ statistic convergances to the normal distribution as the sample size increases and also $Q_{n,p}$ is the strongly consistent estimator of Q_p.

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